Introduction to Sets and Logic (MATH 1190)

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Quiz announcement

The first quiz will be held on Thursday, Oct 16, 9-10 pm in class.

Relevant material is in Chapter 1, excluding those contents that are not covered in the lecture notes (e.g., Section 1.2, the subsection "Applications of Satisfiability" in Section 1.3).

Outline

Outline

Translating from English to logic

Example

If we have

"All men are mortal."
"I am a man."

Doest it follow that "I am mortal?"

Translating from English to logic

Solution.

Let Man(x) denote "x is a man" and Mortal(x) denote "x is mortal". Specify the domain as all people. Then

- the two premises are

$$\forall x(\text{Man}(x) \to \text{Mortal}(x)),$$

$$\text{Man}(\text{I});$$

- the conclusion is Mortal(I).

Is the reasoning correct? □

Translating from English to logic

Example

If we have

“All pigs are mortal.”
“I am mortal.”

Doest it follow that “I am a pig?”

Translating from English to logic

Solution.

Let $\text{Pig}(x)$ denote "x is a pig" and $\text{Mortal}(x)$ denote "x is mortal". Specify the domain as all creatures. Then

- the two premises are

$$\forall x(\text{Pig}(x) \to \text{Mortal}(x)),$$

$$\text{Mortal}(\text{I});$$

- the conclusion is $\text{Pig}(\text{I})$.

Is the reasoning correct? □

Negating quantified expressions

Consider the statement $\forall x P(x)$

- "Every student in our class loves mathematics."

Here $P(x)$ is "x loves mathematics" and the domain is the students in our class.

Negating the statement gives "It is not the case that every student in our class loves mathematics." This implies that

- "There is a student in our class who does not love mathematics."

Symbolically we have that

$$\neg\forall x P(x) \equiv \exists x \neg P(x).$$

Negating quantified expressions

Consider the statement $\exists x P(x)$

- "Some students in our class love mathematics."

Here $P(x)$ is "x loves mathematics" and the domain is the students in our class.

Negating the statement gives "It is not the case that some students in our class love mathematics." This implies that

- "Every student in our class does not love mathematics."

Symbolically we have that

$$\neg \exists x P(x) \equiv \forall x \neg P(x).$$

Negating quantified expressions

The rules for negations for quantifiers

$$\neg \forall x P(x) \equiv \exists x \neg P(x),$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

are called De Morgan's laws for quantifiers.

These two rules are VERY IMPORTANT!

Examples of negating quantified expressions

Example

Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Examples of negating quantified expressions

Proof.

$$\begin{aligned}\neg\forall x(P(x) \to Q(x)) &\equiv \exists x \neg(P(x) \to Q(x)) \\ &\equiv \exists x \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x(\neg\neg P(x) \wedge \neg Q(x)) \\ &\equiv \exists x(P(x) \wedge \neg Q(x)).\end{aligned}$$

□

System specification examples

Example

Translate into predicate logic:

(1) “Every mail message larger than one megabyte will be compressed.”

(2) “If a user is active, at least one network link will be available.”

System specification examples

Solution.

We first decide on predicates and domains (left implicit here) for the variables:

- Let $L(m, y)$ be "Mail message m is larger than y megabytes."
- Let $C(m)$ denote "Mail message m will be compressed."
- Let $A(u)$ represent "User u is active."
- Let $S(n, x)$ represent "Network link n is state x."

Then

(1) $\forall m(L(m, 1) \to C(m))$;

(2) $\exists u A(u) \to \exists n S(n, \text{available})$.

□

Examples from Lewis Carroll

Example

Consider these statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument.

(1) "All lions are fierce."

(2) "Some lions do not drink coffee."

(3) "Some fierce creatures do not drink coffee."

Examples from Lewis Carroll

Solution.

Let $L(x)$, $F(x)$, and $C(x)$ be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

(1) $\forall x(L(x) \to F(x))$.

(2) $\exists x(L(x) \wedge \neg C(x))$.

(3) $\exists x(F(x) \wedge \neg C(x))$.

□

Examples from Lewis Carroll

Example

Consider these statements. The first three are premises and the fourth is the conclusion.

(1) "All hummingbirds are richly colored."

(2) "No large birds live on honey."

(3) "Birds that do not live on honey are dull in color."

(4) "Hummingbirds are small."

Examples from Lewis Carroll

Solution.

Let $Hb(x)$, $L(x)$, $Ho(x)$ and $C(x)$ be the propositional functions "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively.

(1) $\forall x(Hb(x) \to C(x))$.

(2) $\neg\exists x(L(x) \wedge Ho(x))$.

(3) $\forall x(\neg Ho(x) \to \neg C(x))$.

(4) $\forall x(Hb(x) \to \neg L(x))$.

□

Outline

Examples of nested quantifiers

Example

"Every real number has an inverse" is

$$\forall x \exists y (x + y = 0),$$

where the domains of x and y are the real numbers.

This proposition can be viewed as

$$\forall x Q(x),$$

where $Q(x)$ is

$$\exists y P(x, y),$$

where $P(x, y)$ is

$$x + y = 0.$$

Thinking of nested quantification as loops

In the case that the domain consists of finite elements, to see that if $\forall x \forall y P(x, y)$ is true, loop through the values of x:

- At each step, loop through the values for y;
- If for some pair of x and y, $P(x, y)$ is false, then $\forall x \forall y P(x, y)$ is false and both the outer and inner loop terminate.

$\forall x \forall y P(x, y)$ is true if the outer loop ends after stepping through each x.

Thinking of nested quantification as loops

Similarly, in the case that the domain consists of finite elements, to see that if $\forall x \exists y P(x, y)$ is true, loop through the values of x:

- At each step, loop through the values for y;
- The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
- If no y is found such that $P(x, y)$ is true, the outer loop terminates as $\forall x \exists y P(x, y)$ has been shown to be false.

$\forall x \exists y P(x, y)$ is true if the outer loop ends after stepping through each x.

Examples involving the order of quantifiers

Example

Let $P(x, y)$ be the statement "$x + y = y + x$". Assume that the domain is the real numbers. Then

$$\forall x \forall y P(x, y) \quad \text{and} \quad \forall y \forall x P(x, y)$$

have the same truth value.

Examples involving the order of quantifiers

Example

Let $Q(x, y)$ be the statement "$x + y = 0$". Assume that the domain is the real numbers. Then

$$\forall x \exists y P(x, y)$$

is true, but

$$\exists y \forall x P(x, y)$$

is false.

Examples involving the order of quantifiers

Example

Let $P(x, y)$ be the statement

$$\text{“}xy = 0.\text{ ”}$$

Assume that the domain of both x and y consists of the real numbers. What are the truth values of the following propositions?

(1) $\forall x \forall y P(x, y)$;

(2) $\forall x \exists y P(x, y)$;

(3) $\exists y \forall x P(x, y)$;

(4) $\exists x \forall y P(x, y)$;

(5) $\forall y \exists x P(x, y)$;

(6) $\exists x \exists y P(x, y)$.

Examples involving the order of quantifiers

Solution.

(1) False;

(2) True, consider $y = 0$;

(3) True, consider $y = 0$;

(4) True, consider $x = 0$;

(5) True, consider $x = 0$;

(6) True.

□

Examples involving the order of quantifiers

Example

Let $P(x, y)$ be the statement

$$\text{“}\frac{x}{y} = 1.\text{”}$$

Assume that the domain of both x and y consists of the real numbers. What are the truth values of the following propositions?

(1) $\forall x \forall y P(x, y)$;

(2) $\forall x \exists y P(x, y)$;

(3) $\exists y \forall x P(x, y)$;

(4) $\exists x \forall y P(x, y)$;

(5) $\forall y \exists x P(x, y)$;

(6) $\exists x \exists y P(x, y)$.

Examples involving the order of quantifiers

Solution.

(1) False;

(2) False, consider $x = 0$;

(3) False;

(4) False;

(5) False, consider $y = 0$;

(6) True.

$\square$

Quantifications of two variables

$$\forall x \forall y P(x, y)$$

and

$$\forall y \forall x P(x, y)$$

have the same truth value.

They are true if $P(x, y)$ is true for every pair x, y, and they are false if there is a pair x, y for which $P(x, y)$ is false.

Quantifications of two variables

$$\exists x \exists y P(x, y)$$

and

$$\exists y \exists x P(x, y)$$

have the same truth value.

They are true if there is a pair x, y for which $P(x, y)$ is true, and they are false if $P(x, y)$ is false for every pair x, y.

Quantifications of two variables

Neither two of the following propositions

$$\forall x \exists y P(x, y),$$

$$\exists y \forall x P(x, y),$$

$$\exists x \forall y P(x, y),$$

$$\forall y \exists x P(x, y)$$

have the same truth value.

Translating mathematical statements into predicate logic

Example

Translate "The sum of two positive integers is always positive" into a logical expression.

Translating mathematical statements into predicate logic

Solution.

If the domain of both variables consists of all integers, then

$$\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0)).$$

If the domain of both variables consists of all positive integers, then

$$\forall x \forall y (x + y > 0).$$

□

Translating mathematical statements into predicate logic

Example (The limit of a real sequence)

Use quantifiers to express the definition of the limit of a real sequence $\{a_n\}$:

$$\lim_{n\to\infty} a_n = L.$$

Solution.

The definition of

$$\lim_{n\to\infty} a_n = L$$

is: for every real number $\epsilon > 0$, there exists a positive integer N such that, for every positive integer $n > N$, we have $|a_n - L| < \epsilon$. That is,

$$\forall\epsilon > 0\ \exists N\ \forall n((n > N) \to (|a_n - L| < \epsilon)),$$

where the domain of ϵ consists of all real numbers, and the domain of N and n consists of all positive integers. □

Translating nested quantifiers into English

Example

Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y))),$$

where $C(x)$ is "x has a computer,", $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Translating nested quantifiers into English

Solution.

Every student in your school either has a computer or has a friend who has a computer. □

Translating nested quantifiers into English

Example

Translate the statement

$$\exists x \forall y \forall z ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)),$$

where $F(x,y)$ is "x and y are friends," and the domain for both x, y and z consists of all students in your school.

Translating nested quantifiers into English

Solution.

There is a student none of whose friends are also friends with each other. □

Translating English into logical Expressions

Example

Express the statement

"Everyone has exactly one best friend."

Translating English into logical Expressions

Solution.

Let $B(x, y)$ be "y is the best friend of x", and consider the domain of all variables as all people. Then the statement can be expressed as

$$\forall x \exists ! y B(x, y)$$

or

$$\forall x \exists y (B(x, y) \land \forall z ((z \neq y) \to \neg B(x, z)))$$

or

$$\forall x \exists y (B(x, y) \land \forall z (B(x, z) \to (z = y))).$$

□

Translating English into logical Expressions

Example

Express the statement

- “There is a woman who has taken a flight on every airline in the world.”

Translating English into logical Expressions

Solution.

We first decide on predicates and domains (left implicit here) for the variables:

- $P(w, f)$: a woman w has taken a flight f;
- $Q(f, a)$: a flight f is on the airline a.

Then

$$\exists w \forall a \exists f (P(w, f) \land Q(f, a)).$$

□

Negating nested quantifiers

Example

Find the negation of the following statement:

- "There is a woman who has taken a flight on every airline in the world."

Negating nested quantifiers

Solution.

Part 1. Use quantifiers to express the statement:

- "There does not exist a woman who has taken a flight on every airline in the world."

Applying our previous result:

$$\neg\exists w \forall a \exists f (P(w,f) \wedge Q(f,a)).$$

Negating nested quantifiers

Part 2. Use De Morgans Laws to move the negation as far inwards as possible:

$$\begin{aligned}\neg\exists w\forall a\exists f(P(w,f)\wedge Q(f,a)) &\equiv \forall w\neg\forall a\exists f(P(w,f)\wedge Q(f,a))\\ &\equiv \forall w\exists a\neg\exists f(P(w,f)\wedge Q(f,a))\\ &\equiv \forall w\exists a\forall f\neg(P(w,f)\wedge Q(f,a))\\ &\equiv \forall w\exists a\forall f(\neg P(w,f)\vee\neg Q(f,a)).\end{aligned}$$

Negating nested quantifiers

Part 3. Translate the result back into English:

- For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.

Or more succinctly,

- For every woman there is an airline such that this woman has not taken any flight on this airline.

□

Negating nested quantifiers

Example

Find the negation of the following statement:

- “Every Thursday there are some students who do not come to the class MATH 1190.”

Negating nested quantifiers

Solution.

Let $P(x, y)$ be "x comes to the class MATH 1190 on the Thursday of Week y." Then this statement is expressed as

$$\forall y \exists x \neg P(x, y),$$

where the domain of x consists of the students enrolled in the course MATH 1190, and the domain of y consists of all the weeks in the Fall term.

Negating nested quantifiers

Then the negation is

$$\begin{aligned}\neg\forall y\exists x\neg P(x,y) &\equiv \exists y\neg\exists x\neg P(x,y)\\ &\equiv \exists y\forall x\neg\neg P(x,y)\\ &\equiv \exists y\forall x P(x,y),\end{aligned}$$

which means

- There exists a Thursday that all the students come to the class MATH 1190.

□

Negating nested quantifiers

Example (The limit of a real sequence)

Use quantifiers to express that the limit of a real sequence $\{a_n\}$

$$\lim_{n\to\infty} a_n$$

does not exist.

Negating nested quantifiers

Solution.

The negation of

$$\lim_{n\to\infty} a_n = L$$

is expressed as

$$\begin{aligned}
&\neg\forall\epsilon > 0\ \exists N\ \forall n((n > N) \rightarrow (|a_n - L| < \epsilon))\\
\equiv\ &\exists\epsilon > 0\ \neg\exists N\ \forall n((n > N) \rightarrow (|a_n - L| < \epsilon))\\
\equiv\ &\exists\epsilon > 0\ \forall N\ \neg\forall n((n > N) \rightarrow (|a_n - L| < \epsilon))\\
\equiv\ &\exists\epsilon > 0\ \forall N\ \exists n\ \neg(\neg(n > N) \vee (|a_n - L| < \epsilon))\\
\equiv\ &\exists\epsilon > 0\ \forall N\ \exists n\ ((n > N) \wedge (|a_n - L| \geq \epsilon)).
\end{aligned}$$

Negating nested quantifiers

Since $\lim_{n\to\infty} a_n$ does not exist means

$$\lim_{n\to\infty} a_n \neq L$$

for every real number L, this can be expressed as

$$\forall L\, \exists \epsilon > 0\, \forall N\, \exists n\, ((n > N) \land (|a_n - L| \geq \epsilon)),$$

where the domain of L and ϵ consists of all real numbers, and the domain of N and n consists of all positive integers.

That is, for all real number L, there exists a real number $\epsilon > 0$, such that for every positive integer N, there exists a positive integer $n > N$ satisfying $|a_n - L| \geq \epsilon$. □

Recommended exercises

Section 1.4: 34, 39, 43, 45, 50, 51, 53.

Section 1.5: 4, 12, 20, 28, 33, 36, 40.