Introduction to Sets and Logic (MATH 1190)

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Quiz announcement

The first quiz will be held on Thursday, Oct 16, 9-10 pm in class.

Relevant material is in Chapter 1, excluding those contents that are not covered in the lecture notes (e.g., Section 1.2 and 1.8, the subsection "Applications of Satisfiability" in Section 1.3).

Outline

1. Rules of inference
2. Introduction to Proofs

Revisiting the “mortal” example

Example

If we have

“All men are mortal.”
“I am a man.”

How do we get the conclusion “I am mortal” from the premises?

Revisiting the “mortal” example

We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{rl} & \forall x(\text{Man}(x) \rightarrow \text{Mortal}(x)) \\ & \underline{\text{Man}(\text{I})\qquad\qquad\qquad\quad} \\ \therefore & \text{Mortal}(\text{I}) \end{array}$$

We will see shortly that this is a valid argument.

Valid arguments

We will show how to construct valid arguments in two stages. The rules of inference are the essential building block in the construction of valid arguments.

- Propositional Logic:
  Rules of inference
- Predicate Logic:
  Rules of inference for propositional logic plus additional rules to handle variables and quantifiers.

Arguments

Definition

(1) An argument in propositional logic is a sequence of propositions. All but the final proposition are called premises. The last statement is the conclusion.

(2) An argument is valid if the premises imply the conclusion. An argument form is an argument that is valid no matter what propositions are substituted into its propositional variables.

Arguments

It follows immediately from the definition of an argument form that

- the premises $p_1, p_2, \ldots, p_n$ and the conclusion q constitute an argument form if and only if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

  is a tautology.

Rules of inferences

We can always use a truth table to show that an argument form with premises $p_1, p_2, \ldots, p_n$ and an conclusion q is valid, i.e., to show that

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

is a tautology as we did in Section 1.3. However, the truth table becomes gigantic when there are many propositional variables.

Rules of inferences are simple argument forms that will be used to construct more complex argument forms.

Rules of inferences

Modus Ponens

Rule of inference	Tautology
p $\underline{p \to q}$ $\therefore \quad q$	$(p \land (p \to q)) \to q$

Rules of inferences

Example

Let p be "I am learning mathematics."
Let q be "I am happy."

"If I am learning mathematics, then I am happy."
"I am learning mathematics."

"Therefore, I am happy."

Rules of inferences

Modus tollens

	Rule of inference	Tautology
	$\neg q$ $p \to q$	$(\neg q \wedge (p \to q)) \to \neg p$
$\therefore$	$\neg p$	

Rules of inferences

Example

Let p be "I love mathematics."
Let q be "The pigs can fly."

"If I love mathematics, then the pigs can fly."
"The pigs cannot fly."

"Therefore, I do not love mathematics."

Rules of inferences

Hypothetical syllogism

Rule of inference	Tautology
$p \to q$ $q \to r$ $\therefore \quad p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$

Rules of inferences

Example

Let p be "I love mathematics."
Let q be "The pigs can fly."
Let r be "The Phantom Menace would be a timeless classic."

"If I love mathematics, then the pigs can fly."
"If the pigs can fly, then the Phantom Menace would be a timeless classic."

"Therefore, if I love mathematics, then the Phantom Menace would be a timeless classic."

Rules of inferences

Disjunctive syllogism

Rule of inference	Tautology
$p \vee q$ $\neg p$ $\therefore \quad q$	$((p \vee q) \wedge \neg p) \to q$

Rules of inferences

Example

Let p be “Hello Kitty is a cat.”
Let q be “Hello Kitty is a girl.”

“Hello Kitty is a cat a or a girl.”
“Hello Kitty is not a cat.”

“Therefore, Hello Kitty is a girl.”

Rules of inferences

Addition

	Rule of inference	Tautology
	p	$p \to (p \vee q)$
$\therefore$	$p \vee q$	

Rules of inferences

Example

Let p be “Hello Kitty is a girl.”
Let q be “Hello Kitty is a cat.”

“Hello Kitty is a girl.”

“Therefore, Hello Kitty is a girl or a cat.”

Rules of inferences

Simplification

	Rule of inference	Tautology
	$\underline{p \wedge q}$	$(p \wedge q) \rightarrow p$
$\therefore$	p	

Rules of inferences

Example

Let p be "I am a man."
Let q be "I am smart."

"I am a smart man."
(i.e., "I am a man and I am smart.")

"Therefore, I am a man."

Rules of inferences

Conjunction

Rule of inference	Tautology
p q $\therefore \quad p \wedge q$	$(p \wedge q) \rightarrow (p \wedge q)$

Rules of inferences

Example

Let p be "I am a man."
Let q be "I am smart."

"I am a man."
"I am smart."

"Therefore, I am a smart man."

Rules of inferences

Resolution

Rule of inference	Tautology
$p \vee q$ $\neg p \vee r$ $\therefore \quad q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

Rules of inferences

Example

Let p be “I love mathematics.”
Let q and r both be “I will have a math exam.”

“I love mathematics or I will have a math exam.”
“I do not love mathematics or I will have a math exam.”

“Therefore, I will have a math exam.”

Using rules of inference

Example

Show that the premises

- "It is not sunny this afternoon and it is colder than yesterday,"
- "We will go swimming only if it is sunny,"
- "If we do not go swimming, then we will take a canoe trip,"
- "If we take a canoe trip, then we will be home by sunset"

lead to the conclusion

- "We will be home by sunset."

Using rules of inference

Proof.

Let p be “It is sunny this afternoon,”
q “it is colder than yesterday,”
r “We will go swimming,”
s “We will take a canoe trip,”
t “We will be home by sunset.”

Then the premises are

- $\neg p \wedge q$,
- $r \to p$,
- $\neg r \to s$,
- $s \to t$.

The conclusion is simply t.

Using rules of inference

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification
3. $r \to p$	Premise
4. $\neg r$	Modus tollens
5. $\neg r \to s$	Premise
6. s	Modus Ponens
7. $s \to t$	Premise
8. t	Modus Ponens

□

Rules of inference for quantified statements

Rule of inference	Name
$\forall x P(x)$ $\therefore\ P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore\ \forall x P(x)$	Universal genearlization
$\exists x P(x)$ $\therefore\ P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore\ \exists x P(x)$	Existential generalization

Revisiting the “mortal” example

Example

From the premises

- $\forall x(\text{Man}(x) \to \text{Mortal}(x))$,
- Man(I),

we draw the conclusion “Mortal(I)” as:

Step	Reason
1. $\forall x(\text{Man}(x) \to \text{Mortal}(x))$	Premise
2. Man(I) $\to$ Mortal(I)	Universal instantiation
3. Man(I)	Premise
4. Mortal(I)	Modus ponens

Rules of inference for quantified statements

The reasoning in the above example can be simplified by the following rule.

Universal modus ponens

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & Q(a) \end{array}$$

Rules of inference for quantified statements

Similarly, we have the following rule.

Universal transitivity

Suppose that $P(x)$, $Q(x)$ and $R(x)$ have the same domain.

$$\begin{array}{rl} & \forall x(P(x) \to Q(x)) \\ & \forall x(Q(x) \to R(x)) \\ \hline \therefore & \forall x(P(x) \to R(x)) \end{array}$$

Rules of inference for quantified statements

Proof.

Step	Reason
1. $\forall x(P(x) \to Q(x))$	Premise
2. $P(c) \to Q(c)$ for an arbitrary c	Universal instantiation
3. $\forall x(Q(x) \to R(x))$	Premise
4. $Q(c) \to R(c)$ for an arbitrary c	Universal instantiation
5. $P(c) \to R(c)$ for an arbitrary c	Hypothetical syllogism
6. $\forall x(P(x) \to R(x))$	Universal generalization

$\square$

Revisiting the Lewis Carroll example

Example

Consider the premises:

- "All hummingbirds are richly colored."
- "No large birds live on honey."
- "Birds that do not live on honey are dull in color."

How do we get the conclusion "Hummingbirds are small"?

Revisiting the Lewis Carroll example

Solution.

Let $Hb(x)$, $L(x)$, $Ho(x)$ and $C(x)$ be the propositional functions "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively.

Premises:

- $\forall x(Hb(x) \to C(x))$.
- $\neg\exists x(L(x) \wedge Ho(x))$.
- $\forall x(\neg Ho(x) \to \neg C(x))$.

Conclusion:

- $\forall x(Hb(x) \to \neg L(x))$.

Revisiting the Lewis Carroll example

Step	Reason
1. $\neg\exists x(L(x) \wedge Ho(x))$	Premise
2. $\forall x\neg(L(x) \wedge Ho(x))$	De Morgan's law
3. $\forall x(\neg L(x) \vee \neg Ho(x))$	De Morgan's law
4. $\forall x(\neg Ho(x) \vee \neg L(x))$	Commutative law
5. $\forall x(Ho(x) \rightarrow \neg L(x))$	
6. $\forall x(\neg Ho(x) \rightarrow \neg C(x))$	Premise
7. $\forall x(C(x) \rightarrow Ho(x))$	
8. $\forall x(C(x) \rightarrow \neg L(x))$	Universal transitivity
9. $\forall x(Hb(x) \rightarrow C(x))$	Premise
10. $\forall x(Hb(x) \rightarrow \neg L(x))$	Universal transitivity

Outline

Terminologies

- A proof is a valid argument that establishes the truth of a statement.
- A theorem is a statement that can be shown to be true using:
  - definitions,
  - other theorems,
  - axioms,
  - rules of inference.
- A lemma is a helping theorem or a result which is needed to prove a theorem.
- A corollary is a result which follows directly from a theorem.
- Less important theorems are sometimes called propositions.

Terminologies

- A conjecture is a statement that is being proposed to be true. Once a proof of a conjecture is found, it becomes a theorem. It may turn out to be false.

Example (Goldbach's conjecture)

Every even integer greater than 2 can be expressed as the sum of two prime numbers.

Even and odd integers

Definition

- An integer n is even if there exists an integer k such that $n = 2k$.
- An integer n is odd if there exists an integer k such that $n = 2k + 1$.
- Two integers have the same parity if they are both even or both odd; otherwise, they have opposite parity.

Methods of proving theorems

Basic methods of proving theorems include:

- direct proofs;
- indirect proofs:
  - proofs by contraposition;
  - proofs by contradiction.

Examples of direct proofs

Example

Prove that if n is an odd integer, then n^2 is odd.

Examples of direct proofs

Proof.

Assume that n is odd, then $n = 2k + 1$ for an integer k. Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

where $2k^2 + 2k$ is an integer. Thus n^2 is an odd integer. □

Examples of direct proofs

Example

A real number r is rational if there exist integers p and q with $q \neq 0$ such that $r = \frac{p}{q}$. Show that the sum of two rational numbers is rational.

Examples of direct proofs

Proof.

Assume that r and s are rational numbers. Then there must be integers p, q and t, u such that

$$r = \frac{p}{q}, \quad s = \frac{t}{u},$$

where $q \neq 0$ and $u \neq 0$. Then

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu},$$

where $pu + qt$ and qu are integers and $qu \neq 0$. Thus the sum $r + s$ is rational. $\square$

Examples of proofs by contraposition

Example

Show that if n is an integer and n^2 is odd, then n is odd.

Examples of proofs by contraposition

Proof.

Suppose that n is not odd, i.e., n is even, then there exists an integer k such that $n = 2k$. Thus

$$n^2 = 4k^2 = 2(2k^2).$$

Hence n^2 is even, i.e., n^2 is not odd.

Therefore, by contraposition, if n is an integer and n^2 is odd, then n is odd. □

Theorems that are biconditional statements

Example

Let n be an integer. Show that n is odd if and only if n^2 is odd.

Theorems that are biconditional statements

Proof.

We have already shown that

- if n is an odd, then n^2 is odd;
- if n^2 is odd, then n is odd.

Therefore, n is odd if and only if n^2 is odd. □

Examples of proofs by contradiction

Example

Prove that if you pick 22 days from the calendar, at least 4 must fall on the same day of the week.

Examples of proofs by contradiction

Proof.

Assume that no more than 3 of the 22 days fall on the same day of the week. Because there are 7 days in a week, we could only have picked 21 days. This contradicts the assumption that we have picked 22 days. □

A preview of Chapter 4

Example

A real number is called irrational if it is not rational. Show that $\sqrt{2}$ is irrational.

A preview of Chapter 4

Proof.

Suppose that $\sqrt{2}$ is rational, then there exist integers a and b with $\sqrt{2} = \frac{a}{b}$ and $b \neq 0$ such that a and b have no common divisors (will be explained in details in Chapter 4). Then

$$2 = \frac{a^2}{b^2} \quad 2b^2 = a^2.$$

Therefore a^2 must be even.

A preview of Chapter 4

If a^2 is even then a must be even (by a previous example). Thus $a = 2c$ for some integer c, and consequently

$$2b^2 = 4c^2, \quad b^2 = 2c^2.$$

Therefore b^2 is even. Again b must be even as well.
Since both a and b are even, they have a common divisor 2.
This contradicts to our assumption that a and b have no common divisors. Hence $\sqrt{2}$ must be irrational. □

A preview of Chapter 4

Example

Prove that there is no largest prime number.

This proposition is equivalent to

- There are infinitely many prime numbers.

A preview of Chapter 4

Proof.

Assume that there is a largest prime number. Then we can list all the prime numbers $p_1 = 2, p_2 = 3, \ldots, p_n$ from the smallest to the largest p_n. Let

$$r = p_1 \times p_2 \times \cdots \times p_n + 1,$$

then none of the prime numbers $p_1, p_2, \ldots, p_n$ divides r. Therefore, either r is a prime number or there is another prime number q that divides r. The former contradicts to the assumption that p_n is the largest prime number, and the latter contradicts to the assumption that all the prime numbers are in the list $p_1, p_2, \ldots, p_n$.
Therefore, there is no largest prime number. $\square$

Recommended exercises

Section 1.6: 6, 16, 17, 18, 29.

Section 1.7: 6, 10, 11, 12, 13, 24, 28.