Introduction to Sets and Logic (MATH 1190)

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Quiz announcement

The first quiz will be held on Thursday, Oct 16, 9-10 pm in class.

Relevant material is in Chapter 1, excluding those contents that are not covered in the lecture notes (e.g., Section 1.2 and 1.8, the subsection “Applications of Satisfiability” in Section 1.3).

Quiz announcement

Instructions about the quiz:

- This is an open-book quiz. Textbooks and lecture notes are permitted, including paper copies or electronic copies through your tablet or smartphone.
- Sharing materials is not allowed.
- There are two quizzes in this term and each has 50 marks.

Quiz announcement

- Copying answers from others is strictly forbidden. Duplicate quiz papers will BOTH be scored zero.
- For every student who finishes the two quizzes honestly, the sum of your scores in the two quizzes will not be lower than your final exam. For example:
  - You get 90 in the two quizzes, and 30 in the final exam, then the score of your quizzes will be recorded as 90;
  - You get 30 in the two quizzes, and 90 in the final exam, then the score of your quizzes will also be recorded as 90;
  - If two students submit duplicate papers in a quiz, then both of them will get 0 and the zero score cannot be covered by the final exam.

Some exercises

Exercise 12 on Page 65

Let $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements:

n) There are at least two students in your class who have not chatted with the same person in your class.

o) There are two students in the class who between them have chatted with everyone else in the class.

Some exercises

Solution.

n) $\exists x \exists y((x \neq y) \wedge \forall z \neg(C(x,z) \wedge C(y,z)))$.

o) $\exists x \exists y((x \neq y) \wedge \forall z(C(x,z) \vee C(y,z)))$.

$\square$

Some exercises

Exercise 28 on Page 91

Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Outline

1. Sets

Set theory

Set theory is a branch of mathematics that studies the collections of objects.

Set theory is the foundation of mathematics. Modern set theory is one of the most difficult branches in mathematics. (Fortunately we are not going to learn modern set theory.)

Naive set theory

The traditional set theory before the 19th century is called the naive set theory. Compared to modern set theory, naive set theory is defined informally in natural language as below.

Definition

A set is an unordered collection of objects.

- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$.

Russell's paradox

According to this definition of sets, any definable collection is a set. However, a paradox would arise.

Russell's paradox

Let S be the set of all sets that are not members of themselves, i.e.,

$$S = \{x \mid x \notin x\}.$$

Is S an element of itself?

Russell's paradox

If S is not a member of itself, then its definition indicates that it must contain itself.
But if S contains itself, then it contradicts its own definition.

This contradiction is known as the Russell's paradox.
Symbolically:

$$S \in S \leftrightarrow S \notin S.$$

Barber paradox

Russell's paradox has a more popular version.

Barber paradox

The barber is a man in town who shaves all those, and only those, men in town who do not shave themselves. The question is:

- Who shaves the barber?

Barber paradox

If the barber does not shave himself, then he belongs to "those men in town who do not shave themselves", and thus he should shave himself.

If the barber shaves himself, then it contradicts to his principle that he shaves "only those men in town who do not shave themselves".

From naive set theory to modern set theory

In order to get rid of the paradoxes in naive set theory, modern axiomatized set theory was established.

In modern mathematics, we do not try to define sets. Instead, we restrict the terminology "sets" to those who satisfy certain rules (i.e., the axioms). The most popular axiomatic system in modern mathematics is ZFC:

- Zermelo-Fraenkel set theory with the Axiom of Choice: `http://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory`

From naive set theory to modern set theory

For example, one rule is that a set cannot be an element of itself, which helps us avoid the Russell's paradox. In predicate logic, this can be expressed as

$$\forall x \, \neg(x \in x),$$

where the domain of x consists of all sets.

As a corollary, we have that

- The collection of all sets is not a set.

In predicate logic, this can be expressed as

$$\neg\exists x \forall y (y \in x).$$

where the domain of both x and y consists of all sets.

From naive set theory to modern set theory

Since modern set theory (axiomatized set theory) is one of the most difficult branches in mathematics, it will not be the content of our course. It is suggested to read the book given in Page 13 of the lecture note on Sept 11 if you are interested in modern set theory.

Therefore, in our course, we will just consider a set as "a collection of objects" as the ancients did before the 19th century. However, we should have an idea that this is only an informal and non-strict definition. Therefore, in the following we will only deal with the "friendly" sets that would not cause paradoxes.

Describing a set

There are two basic ways of describing a set.

- [Roster method] For example, the set O of odd positive integers less than 10 can be expressed by
$$O = \{1, 3, 5, 7, 9\}.$$
Note that a set is an unordered collection of objects, so
$$O = \{1, 3, 5, 7, 9\} = \{3, 1, 7, 9, 5\}.$$
- [Set builder] For example, the above set O can be expressed by
$$O = \{X \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\},$$
where $\mathbf{Z}^+$ denotes the set of all positive integers.

Some important sets

- $\mathbf{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers;
- $\mathbf{Z} = \{..., -2, -1, 0, 1, 2, \dots\}$, the set of integers;
- $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$, the set of positive integers;
- $\mathbf{Q} = \left\{\frac{p}{q} \,\middle|\, p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\right\}$, the set of rational numbers;
- $\mathbf{R}$, the set of real numbers;
- $\mathbf{R}^+$, the set of positive real numbers;
- $\mathbf{C}$, the set of complex numbers.

Intervals

If a and b are real numbers with $a < b$, we write:

- open interval:

$$(a, b) = \{x \mid a < x < b\};$$

- closed interval:

$$(a, b) = \{x \mid a \leq x \leq b\};$$

- half-closed interval (or half-open interval):

$$[a, b) = \{x \mid a \leq x < b\};$$

$$(a, b] = \{x \mid a < x \leq b\}.$$

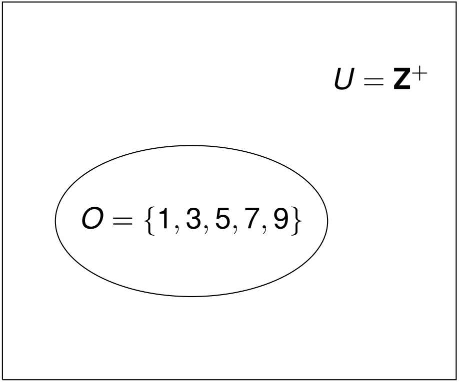
Universal set

The universal set U is the set containing everything currently under consideration.

- Sometimes implicit.
- Sometimes explicitly stated.
- Contents depend on the context.

Venn diagram

A Venn diagram can be used to express the relations between sets. For example:



Empty set

Definition

The empty set is the set with no elements, and is denoted by $\varnothing$ or $\{\}$.

Things to remember

Remark

- Sets can be elements of sets. For example,
$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\},$$
$$\{\{1, 2, 3\}, a, \{b, c\}\}.$$
- The empty set is different from a set containing the empty set, i.e.,
$$\varnothing \neq \{\varnothing\}.$$

Constructing natural numbers

Indeed, in modern set theory, natural numbers are defined recursively as follows:

- $0 = \varnothing$,
- $n + 1 = n \cup \{n\}$.

Explicitly,

$$
\begin{aligned}
&0 = \varnothing,\\
&1 = 0 \cup \{0\} = \varnothing \cup \{\varnothing\} = \{\varnothing\},\\
&2 = 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\},\\
&3 = 2 \cup \{2\} = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\},\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Reference:
http://en.wikipedia.org/wiki/Natural_number

Set equality

Definition

Two sets are equal if and only if they have the same elements.

- Therefore, if A and B are sets, then A and B are equal if and only if
$$\forall x(x \in A \leftrightarrow x \in B).$$
- We write $A = B$ if A and B are equal sets. For example,
$$\{1, 3, 5\} = \{3, 5, 1\} = \{1, 5, 5, 5, 3, 3, 1\}.$$

Subsets

Definition

The set A is a subset of B, or equivalently B is a superset of A, if and only if every element of A is also an element of B, and is denoted by $A \subseteq B$.

That "A is not a subset of B" is denoted by $A \not\subseteq B$.

Subsets

$A \subseteq B$ holds if and only if

$$\forall x (x \in A \to x \in B)$$

is true.

- Because $x \in \varnothing$ is always false, we have that $\varnothing \subseteq S$ for every set S.
- Because $x \in S \to x \in S$, we have that $S \subseteq S$ for every set S.

Proofs related to subsets

We list here the basic methods of proofs related to subsets. They are VERY IMPORTANT and will be frequently used in the future.

- To show that $A \subseteq B$, show that
$$\forall x(x \in A \to x \in B).$$
- To show that $A \not\subseteq B$, show that
$$\exists x(x \in A \wedge x \notin B).$$
- To show that $A = B$, show that $A \subseteq B$ and $B \subseteq A$, i.e.,
$$\forall x(x \in A \leftrightarrow x \in B).$$
More explicitly, to prove $A = B$, we need to show that
$$\forall x((x \in A \to x \in B) \wedge (x \in B \to x \in A)).$$

Proper subsets

If $A \subseteq B$, but $A \neq B$, then we say A is a proper subset of B, denoted by $A \subset B$.

$A \subset B$ if and only if

$$A \subseteq B \wedge B \not\subseteq A,$$

or equivalently,

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Subsets

Example (Exercise 9 on Page 125)

Determine whether each of these statements is true or false.

(1) $0 \in \varnothing$.
(2) $\varnothing \in \{0\}$.
(3) $\{0\} \subset \varnothing$.
(4) $\varnothing \subset \{0\}$.
(5) $\{0\} \in \{0\}$.
(6) $\{0\} \subset \{0\}$.
(7) $\{\varnothing\} \subseteq \{\varnothing\}$.

Subsets

Solution.

(1) False.
(2) False.
(3) False.
(4) True.
(5) False.
(6) False.
(7) True.

$\square$

As a remark, it is logical to say (2) is true if we construct $0 = \varnothing$ as in modern set theory. But this is not a concern of our course on naive set theory.

Subsets

Example

Do there exist sets A and B such that both $A \in B$ and $A \subseteq B$ hold?

Subsets

Solution.

Yes. Let $A = \varnothing$ and $B = \{\varnothing\}$, it is obvious that

$$\varnothing \in \{\varnothing\} \quad \text{and} \quad \varnothing \subseteq \{\varnothing\}.$$

$\square$

Set cardinality

Definition

(1) If there are exactly n distinct elements in a set S, where n is a nonnegative integer, we say that S is finite. Otherwise it is infinite.

(2) The cardinality of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Set cardinality

Example

- $|\{1, 2, 3\}| = 3$.
- Let S be the set of letters of the English alphabet. Then $|S| = 26$.
- $|\varnothing| = 0$.
- $|\{\varnothing\}| = 1$.
- The set of integers is infinite.

Power sets

Definition

The set of all subsets of a set S, denoted $\mathcal{P}(S)$, is called the power set of S.

If a set has n elements, then the cardinality of the power set is 2^n.

Power sets

Example

- $\mathcal{P}(\varnothing) = \{\varnothing\}$.
- $\mathcal{P}(\{a\}) = \{\varnothing, \{a\}\}$. In particular,
$$\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}.$$
- $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. In particular,
$$\mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

Power Sets

Proposition (Exercise 25 on Page 126)

$\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Power Sets

Proof.

"$\to$": Suppose that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, in order to prove $A \subseteq B$, we need to show that

$$x \in A \to x \in B.$$

But $x \in A$ implies that $\{x\} \in \mathcal{P}(A)$, thus $\{x\} \in \mathcal{P}(B)$ because $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Consequently, we obtain $x \in B$.

Power Sets

"$\leftarrow$": Suppose that $A \subseteq B$, in order to prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we need to show that

$$C \in \mathcal{P}(A) \to C \in \mathcal{P}(B),$$

or equivalently,

$$C \subseteq A \to C \subseteq B.$$

Indeed, for each $x \in C$, it follows from $C \subseteq A$ that $x \in A$, and consequently $x \in B$ since $A \subseteq B$. Therefore $C \subseteq B$. $\square$

Power Sets

Corollary (Exercise 22 on Page 126)

$\mathcal{P}(A) = \mathcal{P}(B)$ if and only if $A = B$.

Power Sets

Proof.

"$\to$": Suppose that $\mathcal{P}(A) = \mathcal{P}(B)$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and $\mathcal{P}(B) \subseteq \mathcal{P}(A)$, thus

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

by the previous proposition, and consequently $A = B$.

The "$\leftarrow$" part can be proved in the same way as the "$\to$" part. $\square$

Tuples

Definition

An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is an ordered collection that has a_1 as its first element, and a_2 as its second element, and so on until a_n as its last element.

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if

$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n.$$

Tuples

In particular, ordered 2-tuples are called ordered pairs. The ordered pairs (a, b) and (c, d) are equal if and only if

$$a = c \quad \text{and} \quad b = d.$$

Note that (a, b) and (b, a) are not equal unless $a = b$.

Cartesian products

Definition

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Cartesian products

Example

If $A = \{1, 2\}$, $B = \{a, b, c\}$, then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

and

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

It is obvious that $A \times B \neq B \times A$.

Relations

Definition

A subset R of the Cartesian product $A \times B$, i.e.,

$$R \subseteq A \times B,$$

is called a relation from the set A to the set B.

Relations will be covered in depth in Chapter 9.

Relations

Example

Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. Then a relation

$$R \subseteq A \times B$$

can be used to represent the enrollments of students at the university. That is to say,

$$(x, y) \in R$$

means the student x is enrolled in the course y.

Cartesian products

Definition

The Cartesian products of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1, 2, \ldots, n$. That is to say,

$$\begin{aligned} & A_1 \times A_2 \times \cdots \times A_n \\ = & \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}. \end{aligned}$$

Cartesian products

We use the notation A^n to denote

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

In particular, $A^2 = A \times A$ and $A^3 = A \times A \times A$.

Cartesian products

Example

If we use coordinates to denote the points in a Euclidean plane, then the set of all points in the plane is the cartesian product

$$\mathbf{R}^2 = \{(x, y) \mid x, y \in \mathbf{R}\}.$$

Similarly, $\mathbf{R}^3$ is the Euclidean 3-dimensional space, and more generally, $\mathbf{R}^n$ is the Euclidean n-dimensional space.

Using set notations with quantifiers

We use the statement

$$\forall x \in S(P(x))$$

to denote the quantified statement

$$\forall x P(x),$$

where the domain of x is the set S.

For example,

$$\forall x \in \mathbf{R}(x^2 \geq 0)$$

means

$$\forall x(x^2 \geq 0),$$

where the domain of x consists of all real numbers.

Truth sets of quantifiers

Given a predicate P and a domain D, we define the truth set of P to be the set of elements in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by

$$\{x \in D \mid P(x)\}.$$

For example, the truth set of $|x| = 1$ with the domain consisting of all the integers is the set

$$\{-1, 1\}.$$

Recommended exercises

Section 2.1: 10, 25, 26, 31, 34, 39, 44.