Introduction to Sets and Logic (MATH 1190)

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Oct 16, 2014

Outline

1 Set Operations

A preview of Boolean algebras

Propositional calculus and set theory are both instances of an algebraic system called a Boolean Algebra, which is discussed in Chapter 12.

The operators in set theory are analogous to the corresponding operators in propositional calculus.

In this section, we fix a universal set U and assume that all the considered sets are subsets of U.

A preview of Boolean algebras

Definition

A Boolean Algebra is a set equipped with elements 1 and 0, binary operations $\wedge$ and $\vee$, and a unary operation $\neg$, satisfying these identities:

- $a \wedge 1 = a$, $a \vee 0 = a$;
- $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;
- $a \wedge \neg a = 0$, $a \vee \neg a = 1$;
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
- $a \vee (b \vee c) = (a \vee b) \vee c$;
- $a \wedge (a \vee b) = a$;
- $a \vee (a \wedge b) = a$;
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A preview of Boolean algebras

The following table illustrates the similarity between propositional calculus and set operations.

Boolean algebra	Propositional calculus	Set operations
Elements $a, b, c \ldots$	Propositions $p, q, r \ldots$	Sets $A, B, C \ldots$
1	Tautology **T**	Universal set U
0	Contradiction **F**	Empty set $\varnothing$
$\wedge$	Conjunction $\wedge$	Intersection $\cap$
$\vee$	Disjunction $\vee$	Union $\cup$
$\neg$	Negation $\neg$	Complement $\overline{(\ )}$

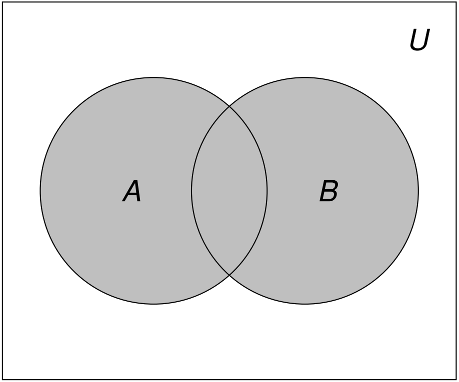
Union

Definition

Let A and B be sets. The union of A and B, denoted by $A \cup B$, is the set

$$\{x \mid x \in A \vee x \in B\}.$$

Union



$A \cup B$ is shaded.

Intersection

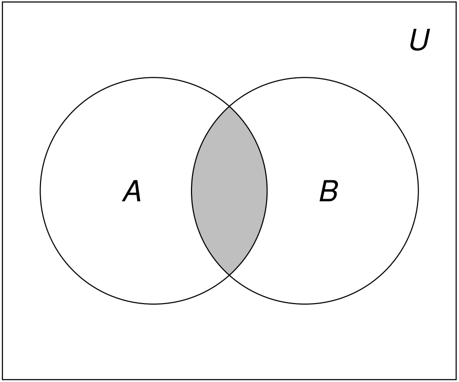
Definition

Let A and B be sets. The intersection of A and B, denoted by $A \cap B$, is the set

$$\{x \mid x \in A \wedge x \in B\}.$$

A and B are said to be disjoint if their intersection is the empty set.

Intersection



$A \cap B$ is shaded.

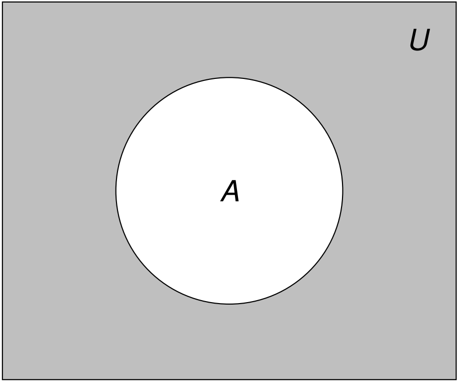
Complement

Definition

Let A be a set. The complement of A (with respect to the universal set U), denoted by $\overline{A}$, or A^c, or $U - A$, is the set

$$\{x \in U \mid x \notin A\}.$$

Complement



$\overline{A}$ is shaded.

Difference

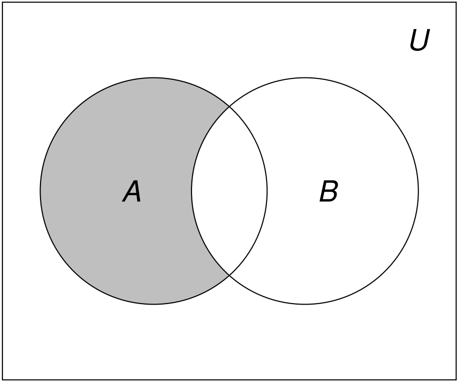
Definition

Let A and B be sets. The difference of A and B, denoted by $A - B$, or $A \setminus B$, is the set

$$A \cap \overline{B} = \{x \mid x \in A \wedge x \notin B\}.$$

Note that $A - B$ and $B - A$ are not the same.

Intersection



$A - B$ is shaded.

Proofs of set identities

Example

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proofs of set identities

Proof.

$$
\begin{aligned}
x \in \overline{A \cap B} &\leftrightarrow x \notin A \cap B \\
&\leftrightarrow x \notin A \ \vee \ x \notin B \\
&\leftrightarrow x \in \overline{A} \ \vee \ x \in \overline{B} \\
&\leftrightarrow x \in \overline{A} \cup \overline{B}.
\end{aligned}
$$

□

Proofs of set identities

Example

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proofs of set identities

Proof.

$$
\begin{aligned}
x \in A \cap (B \cup C) &\leftrightarrow x \in A \ \wedge \ x \in B \cup C \\
&\leftrightarrow x \in A \ \wedge (x \in B \ \vee \ x \in C) \\
&\leftrightarrow (x \in A \ \wedge \ x \in B) \vee (x \in A \ \wedge \ x \in C) \\
&\leftrightarrow x \in A \cap B \ \vee \ x \in A \cap C \\
&\leftrightarrow x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

□

Fundamental set identities

- (Identity laws) $A \cap U = A$, $A \cup \varnothing = A$.
- (Domination laws) $A \cup U = U$, $A \cap \varnothing = \varnothing$.
- (Idempotent laws) $A \cup A = A$, $A \cap A = A$.
- (Complementation law) $\overline{\overline{A}} = A$.
- (Commutative laws) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- (Associative laws) $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
- (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- (De Morgan laws) $\overline{A \cap B} = \overline{A} \cup \overline{B}$, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
- (Absorption laws) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.
- (Negation laws) $A \cup \overline{A} = U$, $A \cap \overline{A} = \varnothing$.

Proofs of set identities

Example

Show that

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}.$$

Proofs of set identities

Proof.

$$\begin{aligned}\overline{A \cap B \cap C} &= \overline{(A \cap B) \cap C} \\ &= \overline{A \cap B} \cup \overline{C} \\ &= (\overline{A} \cup \overline{B}) \cup \overline{C} \\ &= \overline{A} \cup \overline{B} \cup \overline{C}.\end{aligned}$$

□

Generalized union and intersections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets. We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n,$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

These are well defined, since union and intersection are associative.

Generalized union and intersections

Example

For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\} = A_1 = \mathbf{Z}^+,$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

Generalized union and intersections

Let $A_1, A_2, \ldots$ be a sequence of sets. We define:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \cup \ldots,$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots \cap A_n \cap \ldots.$$

Generalized union and intersections

More generally, let I be a set, and A_i a set for each $i \in I$. We define

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I (x \in A_i)\},$$

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I (x \in A_i)\}.$$

Therefore,

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \in \mathbf{Z}^+ (x \in A_i)\},$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \in \mathbf{Z}^+ (x \in A_i)\}.$$

Generalized union and intersections

Example

For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\} = A_1 = \mathbf{Z}^+,$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{i, i+1, i+2, \ldots\} = \varnothing.$$

Generalized union and intersections

Example

For $i = 1, 2, \ldots,$ let $A_i = (0, i)$, i.e., the set of real number x with $0 < x < i$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (0, i) = (0, \infty) = \mathbf{R}^+,$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} (0, i) = (0, 1) = A_1.$$

Recommended exercises

Section 2.2: 14, 16, 19, 24, 31, 48, 50.