Introduction to Sets and Logic (MATH 1190)

Instructor: Lili Shen
Email: shenlili@yorku.ca

Department of Mathematics and Statistics
York University

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Outline

1. Functions
2. Sequences

Functions

Definition

Let A and B be nonempty sets. A function (or map, mapping, transformation)

$$f : A \longrightarrow B$$

is an assignment of each element $a \in A$ to exactly one element of $b = f(a) \in B$.

Functions

For each function $f : A \longrightarrow B$, we define its graph as a subset of $A \times B$, i.e., a relation from A to B, given by

$$\text{Graph } f = \{(a, b) \mid a \in A \text{ and } b \in B \text{ and } b = f(a)\}.$$

On the contrary, a relation $R \subseteq A \times B$ is the graph of a function $f : A \longrightarrow B$, if and only if R contains one, and only one ordered pair (a, b) for every element $a \in A$:

$$\forall x(x \in A \to \exists! y(y \in B \wedge (x, y) \in R)).$$

Therefore, a function $f : A \longrightarrow B$ can also be defined as a relation (satisfying the above requirement) from A to B.

Functions

Given a function $f : A \longrightarrow B$:

- We say f maps A to B or f is a mapping from A to B.
- A is called the domain of f.
- B is called the codomain of f.
- Let $S \subseteq A$. The image of S under f, denoted by $f^{\rightarrow}(S)$, is a subset of B

$$f^{\rightarrow}(S) = \{b \in B \mid \exists s \in S(b = f(s))\}.$$

In particular, $f(A)$ is called the range of f, i.e.,

$$f^{\rightarrow}(A) = \{b \in B \mid \exists a \in A(b = f(a))\}.$$

Functions

- Let $T \subseteq B$. The preimage (or inverse image) of T under f, denoted by $f^{\leftarrow}(T)$, is a subset of A

$$f^{\leftarrow}(T) = \{a \in A \mid f(a) \in T\}.$$

- In particular, if $f(a) = b$, then
  - b is called the image of a under f, i.e.,

$$\{b\} = f^{\rightarrow}(\{a\}).$$

  - a is called a preimage (or an inverse image) of b under f, i.e.,

$$a \in f^{\leftarrow}(\{b\}).$$

- Two functions $f : A \longrightarrow B$ and $g : A' \longrightarrow B'$ are equal if and only if

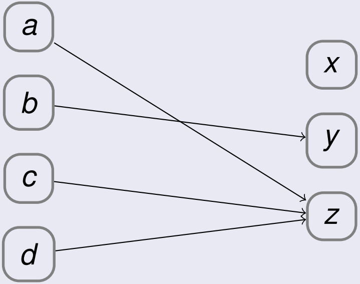
$$A = A' \wedge B = B' \wedge \forall a \in A(f(a) = g(a)).$$

Examples of functions

Example

Consider a function $f : A \longrightarrow B$ with the following assignments:

$$A = \{a, b, c, d\} \qquad B = \{x, y, z\}$$



Examples of functions

Determine:

(1) $f(a)$.
(2) The image of d.
(3) The domain of f.
(4) The codomain of f.
(5) The preimage of $\{x\}$.
(6) The preimage of $\{z\}$.
(7) The image of the subset $\{a, b\} \subseteq A$.
(8) The range of f.

Examples of functions

Solution.

(1) $f(a) = z$.

(2) The image of d is z.

(3) The domain of f is $A = \{a, b, c, d\}$.

(4) The codomain of f is $B = \{x, y, z\}$.

(5) The preimage of $\{x\}$ is $\varnothing$.

(6) The preimage of $\{z\}$ is $\{a, c, d\}$.

(7) The image of the subset $\{a, b\} \subseteq A$ is $\{y, z\}$.

(8) The range of f is $\{y, z\}$.

□

Real-valued functions

A function is called real-valued if the codomain is the set of real numbers. Let $f, g : A \longrightarrow \mathbf{R}$ be two real-valued functions, then $f + g$ and fg are also real-valued functions defined by

$$(f+g)(a) = f(a) + g(a),$$

$$(fg)(a) = f(a)g(a)$$

for all $a \in A$.

Examples of real-valued functions

Example

Let $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ be real-valued functions such that $f(x) = x^2$ and $g(x) = x - x^2$, then

$$(f + g)(x) = f(x) + g(x) = x$$

and

$$(fg)(x) = f(x)g(x) = x^2(x - x^2) = x^3 - x^4.$$

Injections

Definition

A function $f : A \longrightarrow B$ is said to be one-to-one, or injective, or an injection, if

$$\forall a \in A \; \forall b \in A (f(a) = f(b) \rightarrow a = b).$$

Please note that there is a misspelled word in Definition 5 on Page 141 of the textbook:

injunction should be *injection*.

Surjections

Definition

A function $f : A \longrightarrow B$ is said to be onto, or surjective, or a surjection, if

$$\forall b \in B \; \exists a \in A (b = f(a)).$$

Bijections

Definition

A function f is a one-to-one correspondence, or bijective, or a bijection, if it is both one-to-one and onto (injective and surjective).

It is easy to see that, a function $f : A \longrightarrow B$ is bijective if and only if

$$\forall b \in B\ \exists! a \in A (b = f(a)).$$

Examples of real-valued functions

Example

Determine whether the following real-valued functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ are injective or surjective:

(1) $f(x) = x + 1$.

(2) $f(x) = x^2$.

(3) $f(x) = \begin{cases} x + 1, & (x < 0) \\ x - 1. & (x \geq 0) \end{cases}$

(4) $f(x) = e^x$.

Examples of real-valued functions

Solution.

(1) f is injective, since for every $x, y \in \mathbf{R}$, if $f(x) = f(y)$, then

$$x + 1 = y + 1,$$

and it follows that $x = y$.

f is surjective, since for every $y \in \mathbf{R}$, there exists $x = y - 1$ such that

$$f(x) = f(y - 1) = (y - 1) + 1 = y.$$

Thus f is bijective.

Examples of real-valued functions

(2) f is not injective, since there exists $x = -1$ and $y = 1$ such that $x \neq y$, but

$$f(x) = f(-1) = 1 = f(1) = f(y).$$

f is not surjective, since there exists $y = -1$ such that

$$f(x) = x^2 \neq -1$$

for all $x \in \mathbf{R}$.

Examples of real-valued functions

(3) f is not injective, since there exists $x = -1$ and $y = 1$ such that $x \neq y$, but

$$f(x) = f(-1) = 0 = f(1) = f(y).$$

f is surjective, since for every $y \in \mathbf{R}$:

- if $y \geq -1$, then $y + 1 \geq 0$ and

$$f(y+1) = (y+1) - 1 = y;$$

- if $y < -1$, then $y - 1 < 0$ and

$$f(y-1) = (y-1) + 1 = y.$$

Thus for every $y \in \mathbf{R}$, there exits some $x \in \mathbf{R}$ such that $f(x) = y$.

Examples of real-valued functions

(4) f is injective, since for every $x, y \in \mathbf{R}$, if $f(x) = f(y)$, then

$$e^x = e^y,$$

and it follows that

$$x = \ln e^x = \ln e^y = y.$$

f is not surjective, since there exists $y = -1$ such that

$$f(x) = e^x \neq -1$$

for all $x \in \mathbf{R}$. □

Showing that f is injective or surjective

The following is a summary for the methods used in the previous example. Let $f : A \longrightarrow B$ be a function:

- To show that f is injective: Show that for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.
- To show that f is not injective: Show that there exist $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- To show that f is surjective: Show that for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.
- To show that f is not surjective: Show that there exists $y \in B$ such that $f(x) \neq y$ for all $x \in A$.
- To show that f is bijective: Show that f is both injective and surjective.

Inverse functions

Definition

Let $f : A \longrightarrow B$ be a bijection. The inverse function of f is a function $f^{-1} : B \longrightarrow A$ satisfying

$$f^{-1}(b) = a \leftrightarrow f(a) = b$$

for all $b \in B$ and $a \in A$.

A bijection is also called an invertible function because we can define its inverse. A function is not invertible if it is not a bijection.

Examples of inverse functions

Example

In our previous example, $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x + 1$ is invertible, and its inverse is given by

$$f^{-1}(y) = y - 1.$$

Both $f(x) = x^2$, $f(x) = \begin{cases} x - 1, & (x \geq 0) \\ x + 1 & (x < 0) \end{cases}$ and $f(x) = e^x$ are not invertible.

Examples of inverse functions

However, if we restrict the codomain of $f(x) = e^x$ to $\mathbf{R}^+$, i.e., consider it as a function

$$g(x) = e^x : \mathbf{R} \longrightarrow \mathbf{R}^+,$$

then g is a bijection and has a inverse

$$g^{-1}(x) = \ln x : \mathbf{R}^+ \longrightarrow \mathbf{R}.$$

It is noteworthy to point out that

$$f(x) = e^x : \mathbf{R} \longrightarrow \mathbf{R} \quad \text{and} \quad g(x) = e^x : \mathbf{R} \longrightarrow \mathbf{R}^+$$

are different functions, because their codomains are different.

Compositions

Definition

Let $f : A \longrightarrow B$ and $g : C \longrightarrow D$ be functions. If

$$f^{\rightarrow}(A) \subseteq C,$$

Then the composition of g and f is a function $g \circ f : A \longrightarrow D$ satisfying

$$(g \circ f)(a) = g(f(a))$$

for all $a \in A$.

Examples of compositions

Example

Consider functions $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$(f \circ g)(x) = (2x + 1)^2,$$

$$(g \circ f)(x) = 2x^2 + 1.$$

Examples of compositions

Example

Consider functions $f : \mathbf{R}^+ \longrightarrow \mathbf{R}$, $g : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = \ln x$ and $g(x) = 1 + x$, then the composition

$$g \circ f : \mathbf{R}^+ \longrightarrow \mathbf{R}$$

is given by

$$(g \circ f)(x) = 1 + \ln x.$$

But the composition $f \circ g$ does not exist because the range of g is $\mathbf{R}$, which is not a subset of the domain of f.

Examples of compositions

However, if we change the domain of g to $(-1, \infty)$, then the range of the new function

$$g : (-1, \infty) \longrightarrow \mathbf{R}$$

is $\mathbf{R}^+ = (0, \infty)$, and now we have the composition

$$f \circ g : (-1, \infty) \longrightarrow \mathbf{R}$$

as

$$(f \circ g)(x) = \ln(1 + x).$$

Outline

Sequences

Definition

A sequence is a function from a subset of the integers to a set S. Usually we take a function

$$f : \mathbf{N} \longrightarrow S$$

or

$$f : \mathbf{Z}^{+} \longrightarrow S$$

as a sequence, and we use the notation

$$a_n = f(n)$$

to denote the image of the integer n under f. We call each a_n a term of the sequence.

Examples of sequences

Example

Consider the function

$$f : \mathbf{Z}^+ \longrightarrow \mathbf{R}$$

given by

$$f(n) = \frac{1}{n},$$

then we have a real sequence (i.e., a sequence whose terms are all real numbers) $\{a_n\}$ given by

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \dots, a_n = \frac{1}{n}, \dots.$$

Geometric progression

Definition

A geometric progression (or geometric sequence) is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

Geometric progression

A geometric progression can be viewed as restricting the domain of the exponential function

$$f(x) = ar^x : \mathbf{R} \longrightarrow \mathbf{R}$$

to **N**, and thus obtain

$$f(n) = ar^n : \mathbf{N} \longrightarrow \mathbf{R}.$$

Examples of sequences

A real sequence $\{a_n\}$ is a geometric progression if and only if

$$\frac{a_{n+1}}{a_n} = r$$

for some constant r for all terms a_n and a_{n+1}.

Example

The following sequences are all geometric progressions:

- $1, -1, 1, -1, 1, \ldots;$
- $2, 10, 50, 250, 1250, \ldots;$
- $6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$

Arithmetic progression

Definition

An arithmetic progression (or arithmetic sequence) is a sequence of the form

$$a, a+d, a+2d, \dots, a+nd, \dots$$

where the initial term a and the common difference d are real numbers.

Arithmetic progression

An arithmetic progression can be viewed as restricting the domain of the linear function

$$f(x) = a + dx : \mathbf{R} \longrightarrow \mathbf{R}$$

to **N**, and thus obtain

$$f(n) = a + nd : \mathbf{N} \longrightarrow \mathbf{R}.$$

Examples of sequences

A real sequence $\{a_n\}$ is an arithmetic progression if and only if

$$a_{n+1} - a_n = d$$

for some constant d for all terms a_n and a_{n+1}.

Example

The following sequences are both arithmetic progressions:

- $-1, 3, 7, 11, \ldots;$
- $7, 4, 1, -2, \ldots.$

Recurrence relations

- A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms $a_0, a_1, \ldots, a_{n-1}$ of the sequence for all integers $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation. Note that solutions are usually not unique without the initial conditions.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.
- We say that we solved the recurrence relation together with the initial conditions when we find an explicit formula, called a closed formula.

Fibonacci sequence

Definition

The Fibonacci Sequence $\{f_n\}$ is defined by the following recurrence relation and initial conditions.

(1) $f_n = f_{n-1} + f_{n-2}$.

(2) $f_0 = 0$, $f_1 = 1$.

Fibonacci sequence

The closed formula for the Fibonacci Sequence is

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}},$$

which can be obtained by many ways.

Fibonacci sequence

It is easy to calculate that the first several terms of the Fibonacci Sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

We also find that

$$\frac{f_2}{f_1} = \frac{1}{1} = 1, \quad \frac{f_3}{f_2} = \frac{2}{1} = 2,$$
$$\frac{f_4}{f_3} = \frac{3}{2} = 1.5, \quad \frac{f_5}{f_4} = \frac{5}{3} \approx 1.667,$$
$$\frac{f_6}{f_5} = \frac{8}{5} = 1.6, \quad \frac{f_7}{f_6} = \frac{13}{8} = 1.625,$$
$$\ldots\ldots$$

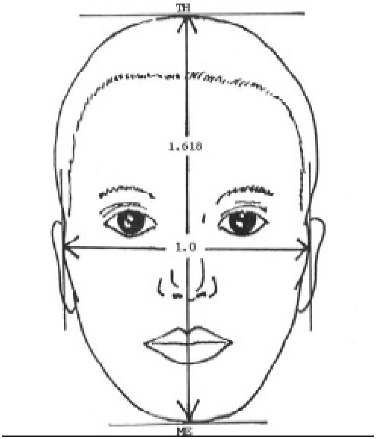
Golden ratio

Indeed, by the knowledge of calculus we have that

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{\sqrt{5}+1}{2} \approx 1.618$$

This number is known as the golden ratio, which is believed as the key to creating aesthetically pleasing art by many artists and architects.

Golden ratio



Golden ratio

Solving recurrence relations

Example

Solve the following recurrence relation and initial condition.

$$\begin{cases} a_n = na_{n-1}, \\ a_1 = 1. \end{cases}$$

Solving recurrence relations

Solution.

$$\begin{aligned} a_n &= na_{n-1} \\ &= n(n-1)a_{n-2} \\ &= \dots \\ &= n(n-1)\dots 2 \cdot a_1 \\ &= n(n-1)\dots 2 \cdot 1 \end{aligned}$$

□

We denote $n! = n(n-1)\dots 2 \cdot 1$ and call it the factorial of n.

Solving recurrence relations

Example

Solve the following recurrence relation and initial condition.

$$\begin{cases} a_n = a_{n-1} + d, \\ a_0 = a. \end{cases}$$

The answer of this example is a closed formula for the terms of an arithmetic progression with initial term $a_0 = a$ and common difference d.

Solving recurrence relations

Solution.

Since $a_n - a_{n-1} = d$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is an arithmetic progression. The solution is

$$\begin{aligned} a_n &= a_{n-1} + d \\ &= a_{n-2} + 2d \\ &= \dots \\ &= a_1 + (n-1)d \\ &= a_0 + nd \\ &= a + nd. \end{aligned}$$

□

Solving recurrence relations

Example

Solve the following recurrence relations and initial conditions.

(1) $$\begin{cases} a_n = a_{n-1} + 3, \\ a_0 = 2. \end{cases}$$

(2) $$\begin{cases} a_n = a_{n-1} + 3, \\ a_1 = 2. \end{cases}$$

Solving recurrence relations

Solution.

(1) Since $a_n - a_{n-1} = 3$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is an arithmetic progression with common difference 3 and initial term $a_0 = 2$. Thus

$$a_n = 2 + 3n.$$

(2) Since $a_n - a_{n-1} = 3$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is an arithmetic progression with common difference 3 and initial term $a_1 = 2$. Thus

$$a_n = 2 + 3(n-1) = 3n - 1.$$

□

Solving recurrence relations

Example

Solve the following recurrence relation and initial condition.

$$\begin{cases} a_n = ra_{n-1}, \\ a_0 = a. \end{cases}$$

The answer of this example is a closed formula for the terms of a geometric progression with initial term $a_0 = a$ and common ratio r.

Solution.

Since $\dfrac{a_n}{a_{n-1}} = r$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is a geometric progression. The solution is

$$\begin{aligned} a_n &= a_{n-1} r \\ &= a_{n-2} r^2 \\ &= \dots \\ &= a_1 r^{n-1} \\ &= a_0 r^n \\ &= ar^n. \end{aligned}$$

□

Solving recurrence relations

Example

Solve the following recurrence relations and initial conditions.

(1) $$\begin{cases} a_n = -a_{n-1}, \\ a_0 = 5. \end{cases}$$

(2) $$\begin{cases} a_n = -a_{n-1}, \\ a_2 = 5. \end{cases}$$

Solving recurrence relations

Solution.

(1) Since $\dfrac{a_n}{a_{n-1}} = -1$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is a geometric progression with common ratio -1 and initial term $a_0 = 5$. Thus

$$a_n = 5 \cdot (-1)^n.$$

(2) Since $\dfrac{a_n}{a_{n-1}} = -1$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is a geometric progression with common ratio -1 and initial term $a_2 = 5$. Thus

$$a_n = 5 \cdot (-1)^{n-2} = 5 \cdot (-1)^n.$$

□

Solving recurrence relations

Example

Solve the following recurrence relation and initial condition.

$$\begin{cases} a_n = 2a_{n-1} - 3, \\ a_0 = -1. \end{cases}$$

Solving recurrence relations

Solution.

From the recurrence relation we have that

$$a_n - 2a_{n-1} = -3, \quad (1)$$
$$a_{n-1} - 2a_{n-2} = -3. \quad (2)$$

and the first two terms are

$$a_0 = -1, \quad a_1 = 2a_0 - 3 = -5.$$

Thus by (1) − (2) we obtain

$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2}).$$

Solving recurrence relations

Thus $a_n - a_{n-1}$ is a geometric progression with common ratio 2 and initial term $a_1 - a_0 = -4$, and consequently

$$a_n - a_{n-1} = -4 \cdot 2^{n-1} = -2^{n+1}. \tag{3}$$

Therefore, by $(3) \cdot 2 - (1)$ we have

$$a_n = -2^{n+2} + 3.$$

$\square$

Solving recurrence relations

Example

Solve the following recurrence relation and initial condition.

$$\begin{cases} a_n = -a_{n-1} + n - 1, \\ a_0 = 7. \end{cases}$$

Solving recurrence relations

Solution.

From the recurrence relation we have that

$$a_n + a_{n-1} = n - 1, \tag{4}$$
$$a_{n-1} + a_{n-2} = n - 2. \tag{5}$$

and the first two terms are

$$a_0 = 7, \quad a_1 = -7.$$

By (4) − (5) we have

$$a_n - a_{n-2} = 1.$$

Solving recurrence relations

This means that the even terms and odd terms of $\{a_n\}$ are respectively an arithmetic progression with common difference 1. Explicitly,

- if $n = 2k$ for some nonnegative integer k, then $\{b_k\} = \{a_{2k}\}$ is an arithmetic progression with common difference 1 and initial term $b_0 = a_0 = 7$, and it follows that
$$a_{2k} = b_k = k + 7.$$
- if $n = 2k + 1$ for some nonnegative integer k, then $\{c_k\} = \{a_{2k+1}\}$ is an arithmetic progression with common difference 1 and initial term $c_0 = a_1 = -7$, and it follows that
$$a_{2k+1} = c_k = k - 7.$$

Solving recurrence relations

Therefore,

$$a_n = \begin{cases} k+7, & (n = 2k) \\ k-7. & (n = 2k+1) \end{cases}$$

Or equivalently,

$$a_n = \begin{cases} \dfrac{n}{2}+7, & (n \text{ is even}) \\ \dfrac{n-15}{2}. & (n \text{ is odd}) \end{cases}$$

□

Recommended exercises

Section 2.3: 6, 22, 28, 39, 69.

Section 2.4: 10, 12, 16.