Introduction to Sets and Logic (MATH 1190)

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Quiz announcement

The second quiz will be held on Thursday, Nov 20, 9-10 pm in class. The contents in our lecture notes from Oct 9 to Nov 13 will be covered. Relevant material in textbook is Section 2.1-2.5 and some sections in Chapter 4 (depending on how much we learn on Nov 13).

Tips for quiz preparation: focus on lecture notes and recommended exercises.

All the rules are the same as Quiz 1. Please check the lecture note on Oct 9 for details.

Outline

Summation notation

Given the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from a real sequence $\{a_n\}$, we use the notation

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum_{m \leq j \leq n} a_j$$

to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

Here, the variable j is called the index of summation, which runs through all the integers starting with its lower limit m and upper limit n.

Summation notation

The choice of the letter j as the index of summation is arbitrary, i.e.,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

More generally, the summation

$$\sum_{j\in S} a_j$$

represents the sum of all a_j for $j \in S$. For example,

$$\sum_{j=m}^{n} a_j = \sum_{j\in\{m,m+1,\ldots,n\}} a_j.$$

Examples of summations

Example

$$\begin{aligned}\sum_{k=4}^{8}(-1)^k &= (-1)^4+(-1)^5+(-1)^6+(-1)^7+(-1)^8\\ &= 1-1+1-1+1\\ &= 1.\end{aligned}$$

Examples of summations

Example

Let $S = \{2, 5, 7, 10\}$, then

$$\sum_{j \in S} j^2 = 2^2 + 5^2 + 7^2 + 10^2 = 4 + 25 + 49 + 100 = 178.$$

Examples of summations

Example (Double summation)

$$
\begin{aligned}
\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i+2i+3i) \\
&= \sum_{i=1}^{4} 6i \\
&= 6+12+18+24 \\
&= 60.
\end{aligned}
$$

Sums of terms of geometric progressions

Theorem

Let a and r be real numbers with $r \neq 0$. Then

$$\sum_{k=0}^{n} ar^k = \begin{cases} \dfrac{ar^{n+1} - a}{r-1}, & (r \neq 1), \\ (n+1)a, & (r = 1). \end{cases}$$

Sums of terms of geometric progressions

Proof.

Let $S_n = \sum_{k=0}^{n} ar^k$. Then

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} + ar^n, && (1) \\ rS_n &= \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n + ar^{n+1}. && (2) \end{aligned}$$

By (2) $-$ (1) we obtain

$$(r-1)S_n = ar^{n+1} - a.$$

Thus $S_n = \dfrac{ar^{n+1} - a}{r-1}$ when $r \neq 1$.

If $r = 1$, it is easy to see that $S_n = (n+1)a$. □

Sums of terms of arithmetic progressions

Theorem

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

(2) *Let a and d be real numbers. Then*

$$\sum_{k=0}^{n}(a+kd) = (n+1)a + \frac{n(n+1)}{2}d.$$

Sums of terms of arithmetic progressions

proof.

(1) Let $S_n = \sum_{k=0}^{n} k$. Then

$$S_n = 1 + 2 + \cdots + (n-1) + n, \tag{3}$$
$$S_n = n + (n-1) + \cdots + 2 + 1. \tag{4}$$

By (3) + (4) we obtain

$$2S_n = n(n+1).$$

Thus $S_n = \dfrac{n(n+1)}{2}$.

Sums of terms of arithmetic progressions

(2)

$$\begin{aligned}\sum_{k=0}^{n}(a+kd) &= \sum_{k=0}^{n} a + \sum_{k=0}^{n} kd \\ &= (n+1)a + d\sum_{k=1}^{n} k \\ &= (n+1)a + \frac{n(n+1)}{2}d.\end{aligned}$$

□

Outline

The size of sets

A natural question in set theory is: how do we compare the size (i.e., the number of elements) of two sets?

Recall that the cardinality of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A. For example:

- $|\{1, 2, 3\}| = 3$.
- Let S be the set of letters of the English alphabet. Then $|S| = 26$.
- $|\varnothing| = 0$.

Therefore, comparing the size of two finite sets is solved: just count the number of elements in each set.

The size of sets

However, how can we compare the size of two infinite sets? In order to achieve this, we need to find another way of comparing the size of sets.

Example

Let A be the set of students in our classroom, and B the set of chairs. Can you tell which set has a larger cardinality, without counting the number of students and chairs?

The size of sets

Solution.

Let every student sit in exactly one chair:

- if every student finds a chair, and there are no spare chairs, then $|A| = |B|$;
- if every student finds a chair, and there are some spare chairs, then $|A| < |B|$;
- if all the chairs are occupied, and some students are standing, then $|A| > |B|$.

□

The size of sets

In the language of set theory, if every student is sitting on exactly one chair, and there are no spare chairs, then we establish a bijection (i.e., a one-to-one correspondence) between the set A of students and the set B of chairs.

In other words, we may prove

"two sets have the same size"

by establishing a bijection between them, without counting their number of elements. This way also works for infinite sets.

Cardinality

Definition

Let A and B be two sets.

- If there is a bijection (i.e., a one-to-one correspondence) from A to B, then we say A and B have the same cardinality, and write $|A| = |B|$, .
- If there is an injection (i.e., a one-to-one function) from A to B, the we say the cardinality of A is less than or the same as the cardinality of B, and write $|A| \leq |B|$.
- If there is an injection from A to B, and there is no bijection between A and B, then we say the cardinality of A is less than the cardinality of B, and write $|A| < |B|$.

Countable sets

Definition

- A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called countable.
- If a set S is countably infinite, we denote the cardinality of S by $|S| = \aleph_0$.
- A set that is not countable is called uncountable.

Countable sets

The terminology “cardinality” is used characterize the number of elements in a set. That is, if two sets have the same cardinality, then they have the same number of elements.

From our intuition, a set has more elements than its proper subset. This is true for finite sets.
However, we must be extremely cautious when referring to infinite sets: an infinite set and its proper subset may have the same cardinality!

Examples of countably infinite sets

Example

Let S be the set of odd positive integers. Then $|S| = \aleph_0$, since

$$f : \mathbf{Z}^{+} \longrightarrow S$$

given by

$$f(n) = 2n - 1$$

is a bijection.

Hilbert's Grand Hotel

Example (Hilbert's Grand Hotel)

Every hotel on earth has only finitely many rooms. If all rooms of a hotel are occupied and a new guest arrives, this guest cannot be accommodated without evicting a current guest.

Now suppose that we have a Grand Hotel with countably infinitely many rooms, each occupied by a guest.

- If a new guest arrives, the manager moves the guest occupying room 1 to room 2, the guest occupying room 2 to room 3 and so on, and fit the newcomer into room 1.

Hilbert's Grand Hotel

- If 10 new guests arrive, the manager moves the guest occupying room 1 to room 11, the guest occupying room 2 to room 12 and so on, and fit the newcomers into the first 10 rooms.
- If a countably infinite number of new guests arrive, the manager moves the person occupying room 1 to room 2, the guest occupying room 2 to room 4, and, in general, the guest occupying room n to room $2n$, and all the odd-numbered rooms (which are countably infinite) will be free for the new guests.

Countable sets

From the definition we know that an infinite set S is countable if and only if there is a bijection

$$f : \mathbf{Z}^{+} \longrightarrow S.$$

Recall that the bijection f exactly defines a sequence (see Page 29 of the lecture note on Oct 23)

$$a_n = f(n), \quad n = 1, 2, \ldots$$

Therefore, S is countable if and only if there exists a sequence $\{a_n\}$, such that every element of S is a term of $\{a_n\}$.

Examples of countably infinite sets

Example

The set of odd positive integers is countable, since we have a sequence

$$1, 3, 5, 7, 9, \ldots$$

that lists all the odd positive integers.

Examples of countably infinite sets

Example

In general, if A is a countable set, then every subset of A is countable. Because we can list the elements of A as (possibly ending after a finite number of terms)

$$a_1, a_2, \dots, a_n, \dots.$$

Every subset $S \subseteq A$ consists of some (or none, or all) of the terms in this sequence, and we can pick them out and list them in the same order as a new sequence. Thus S is also countable.

Examples of countably infinite sets

Example

The set $\mathbf{Z}$ of integers is countable, since we have a sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

that lists all the integers.

Examples of countably infinite sets

Example

The set of rational numbers in the closed interval $[0, 1]$ is countable, since we have a sequence

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots$$

that lists all the rational numbers in $[0, 1]$.

Countable sets

Theorem

If A and B are countable sets, then so is $A \cup B$.

Countable sets

Proof.

There are three cases:

- If A and B are both finite, then $A \cup B$ is also finite, thus countable.
- If one of A and B is countably infinite, suppose A can be listed in an infinite sequence $a_1, a_2, \dots, a_n, \dots$ and B has finitely many elements $b_1, b_2, \dots, b_m$, then we can list the elements of $A \cup B$ as

$$b_1, b_2, \dots, b_m, a_1, a_2, \dots, a_n, \dots.$$

Countable sets

- If both A and B are countably infinite, then we can list the elements of A and B respectively as

$$a_1, a_2, \ldots, a_n, \ldots,$$

$$b_1, b_2, \ldots, b_n, \ldots.$$

Therefore, we can list the elements of $A \cup B$ as

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots.$$

□

Countable sets

Theorem

If for each $i \in \mathbf{Z}^+$, A_i is a countable set, then

$$\bigcup_{i=1}^{\infty} A_i$$

is a countable set.

Countable sets

Proof.

We only prove the case that each A_i ($i \in \mathbf{Z}^+$) is countably infinite. In this case, we can list the elements of each A_i as

$$a_{i1}, a_{i2}, \ldots, a_{in}, \ldots.$$

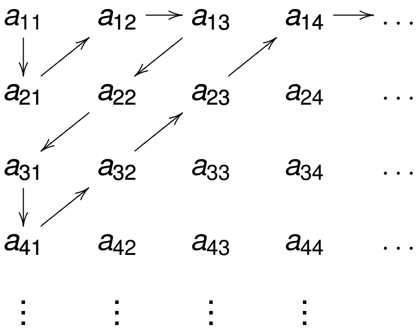
Therefore, we can list the elements of $\bigcup_{i=1}^{\infty} A_i$ as

$$a_{11}, a_{21}, a_{12}, a_{13}, a_{22}, a_{31}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots.$$

□

Countable sets

The following diagram illustrates the listing of elements in $\bigcup_{i=1}^{\infty} A_i$ in the last proof:



Examples of countable sets

Example

We have known that the rational numbers in $[0, 1]$ is countable. Similarly, the rational numbers in any closed interval $[n, n+1]$ with $n \in \mathbf{Z}$ is countable. Therefore, the set $\mathbf{Q}$ of all rational numbers

$$\mathbf{Q} = \bigcup_{n \in \mathbf{Z}} \{q \mid q \text{ is a rational number in } [n, n+1]\}$$

is countable.

Surprisingly, the set of $\mathbf{Q}$ of rational numbers has as many elements as the set $\mathbf{Z}^+$ of positive integers!

Examples of uncountable sets

Example

The set **R** of real numbers is uncountable.

We shall use the famous Cantor diagonalization argument to prove it.

Examples of uncountable sets

Proof.

It suffices to show that the open interval $(0, 1)$ is uncountable. Now we write every real number in $(0, 1)$ as an infinite decimal:

$$r = 0.a_1 a_2 a_3 \ldots,$$

where $a_i \in \{0, 1, 2, \ldots, 9\}$. Suppose that we can list the real numbers in $(0, 1)$ as

$$r_1, r_2, r_3, \ldots, r_n, \ldots,$$

let the decimal representation of these real numbers be

Examples of uncountable sets

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\
&\vdots
\end{aligned}
$$

Then there is a real number $r = 0.a_1a_2a_3\ldots$ given by

$$
a_i = \begin{cases} 1, & \text{if } d_{ii} \neq 1; \\ 2, & \text{if } d_{ii} = 1. \end{cases}
$$

Examples of uncountable sets

Note that $r \in (0, 1)$, but

$$\forall i \in \mathbf{Z}^+ (r \neq r_i),$$

since $a_i \neq d_{ii}$. This contradicts to our hypothesis that $\{r_n\}$ lists all the real numbers in $(0, 1)$. □

Examples of uncountable sets

Example

The set S of irrational numbers is uncountable. Otherwise,

$$\mathbf{R} = S \cup \mathbf{Q}$$

would be countable since $\mathbf{Q}$ is, contradicting to the fact that $\mathbf{R}$ is uncountable.

Examples of uncountable sets

This example shows that irrational numbers are “more” than rational numbers. Indeed, much more than you think:

From the viewpoint of measure theory, if you pick a random point in a real line, then

- the probability that the point is a rational number is 0%, and
- the probability that the point is a irrational number is 100%!

In other words, we could say:

“Almost all real numbers are irrational.”

Schröder-Bernstein theorem

It is usually not easy to prove that two sets have the same cardinality by establishing a bijection between them. The following theorem provides a more efficient way.

Theorem (Schröder-Bernstein)

If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.
In other words, if there are injection $f: A \longrightarrow B$ and injection $g: B \longrightarrow A$, then there is a bijection between A and B.

Schröder-Bernstein theorem

Proof.

Without loss of generality, we may suppose that A and B are disjoint. Otherwise, one just needs to take $A' = A \times \{0\}$ and $B' = B \times \{1\}$, and prove that the disjoint sets A' and B' have the same cardinality, since obviously $|A| = |A'|$ and $|B| = |B'|$.

Now suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are injections. For each $x_0 \in X$, by applying f we get $y_0 = f(x_0)$, and by applying g to y_0 we get $x_1 = g(y_0)$, and by applying f again to x_1 we get $y_1 = f(x_1)$, and so on. Then we have a sequence

$$x_0, y_0, x_1, y_1, \dots, x_n, y_n, \dots,$$

where $y_n = f(x_n)$ and $x_{n+1} = g(y_n)$.

Schröder-Bernstein theorem

Since both f and g are injections, if an element appears twice in the above sequence, then the first one must be x_0, i.e., $g(y_n) = x_0$ for some n with $x_0, x_1, \ldots, x_n$ different from each other and $y_0, y_1, \ldots, y_n$ different from each other.

If there no duplicate elements, consider if there is $y_{-1} \in Y$ satisfying $g(y_{-1}) = x_0$, which must be unique if exists. Similarly, we may search for x_{-1} so that $y_{-1} = f(x_{-1})$, and so on.

By repeating these procedures we have the following four types of sequences:

Schröder-Bernstein theorem

- Type I: cyclic sequence

$$x_0 \xrightarrow{f} y_0 \xrightarrow{g} x_1 \xrightarrow{f} y_1 \xrightarrow{g} \dots \xrightarrow{g} x_n \xrightarrow{f} y_n \xrightarrow{g} x_0$$

- Type II: two-sided infinite sequence

$$\dots \xrightarrow{g} x_{-1} \xrightarrow{f} y_{-1} \xrightarrow{g} x_0 \xrightarrow{f} y_0 \xrightarrow{g} x_1 \xrightarrow{f} y_1 \xrightarrow{g} \dots$$

- Type III: one-sided infinite sequence (x_0 has no preimage under g)

$$x_0 \xrightarrow{f} y_0 \xrightarrow{g} x_1 \xrightarrow{f} y_1 \xrightarrow{g} \dots \xrightarrow{g} x_n \xrightarrow{f} y_n \xrightarrow{g} \dots$$

Schröder-Bernstein theorem

- Type IV: one-sided infinite sequence (y_0 has no preimage under f)

$$y_0 \xrightarrow{g} x_0 \xrightarrow{f} y_1 \xrightarrow{g} x_1 \xrightarrow{f} \dots \xrightarrow{f} y_n \xrightarrow{g} x_n \xrightarrow{f} \dots$$

Since f and g are injections, each element of X and Y appears in exactly one of these sequences. Therefore, mapping x_n in each sequence to the corresponding y_n ($n \in \mathbf{Z}$), we obtain a bijection from X to Y. □

Proofs of cardinality

Example

Show that $|(0,1)| = |[0,1]|$.

Proofs of cardinality

Proof.

Let $f : (0,1) \longrightarrow [0,1]$ be

$$f(x) = x,$$

and $g : [0,1] \longrightarrow (0,1)$ be

$$g(x) = \frac{x+1}{3}.$$

Then both f and g are injections. Thus $|(0,1)| = |[0,1]|$. □

Proofs of cardinality

Example

Let A and B be two sets. Show that if $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Proofs of cardinality

Hint.

Since $|A| = |B|$, there is a bijection $f : A \longrightarrow B$. Show that $g : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ given by

$$\forall S \subseteq A,\ g(S) = f^{\rightarrow}(S) = \{f(a) \mid a \in S\},$$

is a bijection. □

Recommended exercises

Section 2.4: 30, 32, 34, 35, 37.

Section 2.5: 10, 11, 16, 18, 19, 20, 33.