Math 1190 B: Final Exam (2014 Fall)

(Time: 180 minutes, Saturday, December 13, 2014, 7:00-10:00 pm)

NAME (print): ____
(Family) (Given)

SIGNATURE: ____

STUDENT NUMBER: ____

Instructions:

1. Aids permitted: This is a half-open-book exam. Everyone is allowed to take part in the exam with one A4 sheet of paper, and anything can be written or printed on both sides of this paper. No other aids are permitted, including (but not limited to) a second sheet of paper, textbooks, lecture notes, calculators and electronic devices.
2. Check that you have all 9 pages (including this title page). There are 2 additional blank pages that can be torn off as scratch papers.
3. Clearly indicate which part of the question you are answering in the space provided. You may write the answers on the back of each page if you need more space.

FOR INSTRUCTOR'S USE ONLY

1	2	3	4	5	6	7	8	Total

1. (*20 marks*) Suppose that you are the manager of Hilbert's Grand Hotel:

(i) your hotel has countably infinitely many rooms;

(ii) each room is already occupied by a guest;

(iii) two or more guests cannot be accommodated in one room.

Now a group of new guests arrive at your hotel. Denote the set of new guests by A and its cardinality by $|A|$. Consider the following questions: if your answer is YES, explain how you accommodate the new guests; if your answer is NO, specify the reason.

(a) If $|A| = 5$, is it possible to accommodate all the new guests without evicting a current guest?

(b) If $|A| = |\mathbf{Q}|$ ($\mathbf{Q}$ is the set of rational numbers), is it possible to accommodate all the new guests without evicting a current guest?

(c) If $|A| = |\mathbf{R}|$ ($\mathbf{R}$ is the set of real numbers), is it possible to accommodate all the new guests without evicting a current guest?

(d) If $|A| = |\mathbf{R}|$ and you are allowed to accommodate a countably infinite number of guests in each room regardless of the rule (iii), is it possible to accommodate all the new guests without evicting a current guest?

Solution. (a) Yes. We can move the guest occupying room n to room $n+5$, and the first 5 rooms will be free for the new guests. (5 marks)

(b) Yes. Since $\mathbf{Q}$ is a countably infinite set, we can move the guest occupying room n to room $2n$, and all the odd-numbered rooms (which are countably infinite) will be free for the new guests. (5 marks)

(c) No. Because $\mathbf{R}$ is uncountable, and we only have countably many rooms. (5 marks)

(d) No. If for each $i \in \mathbf{Z}^+$, A_i is a countable set, then $\bigcup_{i=1}^{\infty} A_i$ is still a countable set. Therefore even if we are allowed to accommodate a countably infinite number of guests in each room regardless of the rule (iii), it is impossible to accommodate uncountably many guests. (5 marks)

□

2. (*15 marks*)

(a) Find the quotient and remainder when -100 is divided by 11.

(b) In the poset $(\mathbf{Z}^+, |)$, give an example of comparable elements and an example of incomparable elements.

(c) Describe the congruence class of 2 modulo 5.

Solution. (a) Since $-100 = 11 \cdot (-10) + 10$, the quotient is -10 and the remainder is 10. (5 marks)

(b) For example, $(1, 2)$ is a comparable pair of elements in $(\mathbf{Z}^+, |)$, and $(2, 3)$ is an incomparable pair of elements in $(\mathbf{Z}^+, |)$. (5 marks)

(c) $[2]_5 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ or $[2]_5 = \{5k + 2 \mid k \in \mathbf{Z}\}$. (5 marks)

$\square$

3. (*15 marks*) Using quantifiers, logical operators (except the negation $\neg$) and mathematical operators, translate each of following statements into a logical expression.
(Note: You may use the negation $\neg$ in an intermediate step, but it is not allowed to include $\neg$ in the final answer. For example, "$\neg(A \subseteq B)$" is not considered as an answer of (b).)

(a) Let $f : A \longrightarrow B$ be a function between sets. Translate the statement

"f is **not** surjective."

(b) Let A and B be sets. Translated the statement "$A \not\subseteq B$", i.e.,

"A is **not** a subset of B."

(c) Let a and b be positive integers. Translate the statement "$a \nmid b$", i.e.,

"a does **not** divide b."

Solution. (a) $\exists b \in B \ \forall a \in A(f(a) \neq b)$. (5 marks)

(b) $\exists a \in A(a \notin B)$. (5 marks)

(c) $\forall c \in \mathbf{Z}(b \neq ac)$. (5 marks)

$\square$

4. (*10 marks*)

(a) Find the prime factorizations of 84 and 120.

(b) Using the prime factorizations above, find the least common multiple of 84 and 120.

Solution. (a) $84 = 2^2 \cdot 3 \cdot 7, \quad 120 = 2^3 \cdot 3 \cdot 5.$ (5 marks)

(b) $\operatorname{lcm}(84, 120) = 2^3 \cdot 3 \cdot 5 \cdot 7 = 840.$ (5 marks)

$\square$

5. (*10 marks*)

(a) Find the real sequence $\{a_n\}$ that satisfies the following recurrence relation and initial condition:

$$\begin{cases} a_n = a_{n-1} + 3, \\ a_1 = 2. \end{cases}$$

(b) Show that

$$\sum_{i=1}^{n} a_i = \frac{n(3n+1)}{2}.$$

Solution. (a)

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= a_{n-2} + 2 \cdot 3 \\ &= \dots \\ &= a_1 + (n-1) \cdot 3 \\ &= 2 + 3(n-1) \\ &= 3n - 1. \end{aligned} \qquad \text{(5 marks)}$$

(b) Let $P(n)$ denote "$\sum_{i=1}^{n} a_i = \frac{n(3n+1)}{2}$."

For $n = 1$, $P(1)$ is true since $\frac{1(3 \cdot 1 + 1)}{2} = 2 = a_1$.

Assume the $P(k)$ is true, i.e., $\sum_{i=1}^{k} a_i = \frac{k(3k+1)}{2}$. We need to show that $P(k+1)$ is true, i.e., $\sum_{i=1}^{k+1} a_i = \frac{(k+1)(3(k+1)+1}{2} = \frac{(k+1)(3k+4)}{2}$. Indeed,

$$\begin{aligned} \sum_{i=1}^{k+1} a_i &= \left(\sum_{i=1}^{k} a_i \right) + 3(k+1) - 1 \\ &= \frac{k(3k+1)}{2} + (3k+2) \\ &= \frac{k(3k+1) + 2(3k+2)}{2} \\ &= \frac{3k^2 + 7k + 4}{2} \\ &= \frac{(k+1)(3k+4)}{2}, \end{aligned}$$

the conclusion thus follows. (5 marks)

$\square$

Alternative solution for (a). Since $a_n - a_{n-1} = 3$ for all $n \in \mathbf{Z}^+$, $\{a_n\}$ is an arithmetic progression with common difference 3 and initial term $a_1 = 2$. (2 marks) Thus

$$a_n = 2 + 3(n-1) = 3n - 1. \quad \text{(3 marks)}$$

□

Alternative solution for (b). Let $S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}(3i-1)$. Then

$$S_n = 2 + 5 + \cdots + (3n-4) + (3n-1), \tag{1}$$
$$S_n = (3n-1) + (3n-4) + \cdots + 5 + 2. \tag{2}$$

(2 marks) By (1)+(2) we obtain

$$2S_n = (3n+1) + (3n+1) + \cdots + (3n+1) + (3n+1) = n(3n+1).$$

Therefore

$$S_n = \frac{n(3n+1)}{2}. \quad \text{(3 marks)}$$

□

Another alternative solution for (b). Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, (2 marks) it follows that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n}(3i-1) = 3\Big(\sum_{i=1}^{n} i\Big) - n = 3 \cdot \frac{n(n+1)}{2} - n = \frac{3n(n+1) - 2n}{2} = \frac{n(3n+1)}{2}.$$

(3 marks)

□

6. (*10 marks*) Let A be the closed interval

$$A = [-1, 1] = \{x \in \mathbf{R} \mid -1 \le x \le 1\},$$

and B the open interval

$$B = (-1, 1) = \{x \in \mathbf{R} \mid -1 < x < 1\}.$$

Use the Schröder-Bernstein theorem to that A and B have the same cardinality.

Proof. Let $f : A \longrightarrow B$ be

$$f(x) = \frac{x}{2}, \quad \text{(3 marks)}$$

and $g : B \longrightarrow A$ be

$$g(x) = x. \quad \text{(3 marks)}$$

Then both f and g are injections. (2 marks) Thus $|A| = |B|$. (2 marks) $\square$

7. (*10 marks*) Show that every amount of postage of 8 cents or more can be formed using just 3-cent and 5-cent stamps.

Proof. Let $P(n)$ denote "a postage of n cents can be formed using just 3-cent and 5-cent stamps."

For $n = 8$, $P(8)$ is true since $8 = 3 + 5$;

For $n = 9$, $P(9)$ is true since $9 = 3 + 3 + 3$;

For $n = 10$, $P(10)$ is true since $10 = 5 + 5$. (5 marks)

Assume $P(j)$ is true for all integers j with $8 \leq j \leq k$, where we assume $k \geq 10$, we need to show that $P(k + 1)$ is true.

In order to form $k + 1$ cents of postage, note that $P(k - 2)$ is true (since $k \geq 10$) by our inductive hypothesis. Put one more 3-cent stamp on the evelope with $k - 2$ cents of postage, and we have formed $k + 1$ cents of postage, as desired.

Hence, $P(n)$ is true for all integers $n \geq 8$. (5 marks) □

8. (*10 marks*) Find all solutions to the system of congruences

$$\begin{cases} x \equiv 2 \pmod 3, \\ x \equiv 1 \pmod 4, \\ x \equiv 4 \pmod 7. \end{cases}$$

Solution 1. Let $m = 3 \cdot 4 \cdot 7 = 84$, $M_1 = 28$, $M_2 = 21$, $M_3 = 12$. (2 marks) By inspection we find that

$$\begin{aligned} 28 \cdot 1 &\equiv 1 \pmod 3, \\ 21 \cdot 1 &\equiv 1 \pmod 4, \\ 12 \cdot 3 &\equiv 1 \pmod 7. \end{aligned} \qquad \text{(3 marks)}$$

Thus the integers x satisfying

$$x \equiv 2 \cdot 28 \cdot 1 + 1 \cdot 21 \cdot 1 + 4 \cdot 12 \cdot 3 = 221 \equiv 53 \pmod{84}$$

are the solutions. (5 marks) □

Solution 2. If $x \equiv 2 \pmod 3$, then $x = 3t + 2$, where $t \in \mathbf{Z}$. (2 marks) Substituting this expression for x into the second congruence, we get

$$3t + 2 \equiv 1 \pmod 4,$$

and consequently $3t \equiv -1 \equiv 3 \pmod 4$. Since 3 is an inverse of 3 modulo 4, we multiply both sides by 3 and obtain

$$9t \equiv 9 \pmod 4,$$

and consequently $t \equiv 9 \equiv 1 \pmod 4$, which means $t = 4u + 1$, where $u \in \mathbf{Z}$, and thus $x = 3(4u + 1) + 2 = 12u + 5$. (3 marks)

Substituting this expression for x into the third congruence, we get

$$12u + 5 \equiv 4 \pmod 7,$$

and consequently $12u \equiv -1 \equiv 6 \pmod 7$. Since 3 is an inverse of 12 module 7, we multiply both sides by 3 and obtain

$$36u \equiv 18 \pmod 7,$$

and consequently $u \equiv 18 \equiv 4 \pmod 7$, which means $u = 7v + 4$, where $v \in \mathbf{Z}$, and thus

$$x = 12(7v + 4) + 5 = 84v + 53.$$

Therefore, the integers x satisfying $x \equiv 53 \pmod{84}$ are the solutions. (5 marks) □