Math 1190 B: Quiz 2 (2014 Fall)

(Time: 60 minutes, Thursday, November 20, 2014, 9:00-10:00 pm)

NAME (print): ____
(Family) (Given)

SIGNATURE: ____

STUDENT NUMBER: ____

Instructions:

1. Time allowed: 60 minutes.
2. Aids permitted: This is an open-book quiz. Textbooks and lecture notes are permitted.
3. Check that you have all 5 pages (including this title page).
4. Clearly indicate which part of the question you are answering in the space provided. If you need more space, be sure to indicate clearly where the rest of your answer is to be found.
5. Marks for each question are as indicated. You should allocate your time correspondingly.
6. Your work must justify the answer you give.
7. Sharing materials is not allowed.

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1	/12
2	/10
3	/16
4	/12
Total	/50

Question 1 (*12 marks*)**.** Read the following statements:

(i) $\varnothing \in \{\{\varnothing\}\}$.

(ii) If two sets A, B satisfy $A - B = \varnothing$, then $A = B$.

(iii) If $A = \{n \mid n \text{ is an even integer and } n \geq 2014\}$ and $\mathbf{Q}$ is the set of rational numbers, then $|A| < |\mathbf{Q}|$.

(iv) When -100 is divided by 9, the quotient is -11 and the remainder is -1, since

$$-100 = 9(-11) - 1.$$

Please determine and mark True (T) or False (F) for each statement:
(You do not need to write down the reason)

(i)	(ii)	(iii)	(iv)
F	F	F	F

Hint. (3 marks for each answer)

(i) $\{\{\varnothing\}\}$ is a set with only one object $\{\varnothing\}$, so $\varnothing$ is not an element of $\{\{\varnothing\}\}$.

(ii) Let $A = \varnothing$ and $B = \{0\}$, then $A \neq B$ and $A - B = \varnothing$.

(iii) Both A and $\mathbf{Q}$ are countably infinite sets and therefore $|A| = |\mathbf{Q}| = \aleph_0$.

(iv) The remainder cannot be negative. So the quotient of -100 divided by 9 is -12 and the remainder is 8, since

$$-100 = 9(-12) + 8.$$

□

Question 2 (*10 marks*)**.** Find the value of

$$\sum_{i\in S}\sum_{j\in T} ij,$$

where $S = \{1, 2, 4\}$, $T = \{1, 3\}$.

Solution 1.

$$\begin{aligned}
\sum_{i\in S}\sum_{j\in T} ij &= \sum_{i\in S}(i + 3i) \\
&= \sum_{i\in S} 4i && \text{(5 marks)} \\
&= 4 + 8 + 16 \\
&= 28. && \text{(5 marks)}
\end{aligned}$$

□

Solution 2.

$$\begin{aligned}
\sum_{i\in S}\sum_{j\in T} ij &= \sum_{j\in T}(j + 2j + 4j) \\
&= \sum_{j\in T} 7j && \text{(5 marks)} \\
&= 7 + 21 \\
&= 28. && \text{(5 marks)}
\end{aligned}$$

□

Question 3 (*16 marks*)**.** Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a real-valued function given by

$$f(x) = \begin{cases} x - 1, & (x < 0) \\ x + 1. & (x \geq 0) \end{cases}$$

(i) Determine whether f is an injection (i.e., a one-to-one function). Justify your answer.

(ii) Determine whether f is a surjection (i.e., an onto function). Justify your answer.

Solution. (i) f is an injection. (3 marks)

Note that the range of f is $(-\infty, -1) \cup [1, \infty)$. Suppose $f(a) = f(b)$ for some $a, b \in \mathbf{R}$, we need to prove $a = b$. Indeed,

- if $f(a) = f(b) < -1$, then

$$a - 1 = f(a) = f(b) = b - 1,$$

  and consequently $a = b$;

- if $f(a) = f(b) \geq 1$, then

$$a + 1 = f(a) = f(b) = b + 1,$$

  and consequently $a = b$. (5 marks)

(ii) f is not a surjection. (3 marks)

Since the range of f is $(-\infty, -1) \cup [1, \infty)$, there exists $0 \in \mathbf{R}$ such that $f(x) \neq 0$ for all $x \in \mathbf{R}$. (5 marks) □

Question 4 (*12 marks*).

(i) Find the greatest common divisor $\gcd(30, 78)$.

(ii) Express $\gcd(30, 78)$ as a linear combination of 30 and 78.

Solution. (i) We use the Euclidean algorithm to find $\gcd(30, 78)$:

$$\begin{aligned}78 &= 30 \cdot 2 + 18,\\ 30 &= 18 \cdot 1 + 12,\\ 18 &= 12 \cdot 1 + 6,\\ 12 &= 6 \cdot 2.\end{aligned}$$

Therefore, $\gcd(30, 78) = 6$. (6 marks)

(ii)

$$\begin{aligned}6 &= 18 - 12\\ &= 18 - (30 - 18)\\ &= 2 \cdot 18 - 30\\ &= 2 \cdot (78 - 30 \cdot 2) - 30\\ &= 2 \cdot 78 - 5 \cdot 30. \qquad \text{(6 marks)}\end{aligned}$$

□

Alternative solution for (i).

$$\begin{aligned}30 &= 2 \cdot 3 \cdot 5, \text{ (2 marks)}\\ 78 &= 2 \cdot 3 \cdot 13, \text{ (2 marks)}\end{aligned}$$

thus $\gcd(30, 78) = 2 \cdot 3 = 6$. (2 marks) □