Fundamentals of Mathematics (MATH 1510)

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September 14-18, 2015

Outline

Integer exponents

Definition

For $a \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, the *n*th power of a is defined as

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

The number a is called the base, and n is called the exponent.

For $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{Z}^+$,

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

Integer exponents

The operation in the form of a^n is called exponentiation.

a^n usually reads as

- "a (raised) to the nth (power)",
- "a (raised) to the (power of) n", or
- "a (raised) by the exponent of n",

where the words in the brackets can be omitted.

In particular, a^2 reads as "a square", and a^3 reads as "a cube".

Laws of exponents

Proposition

(1) $a^m a^n = a^{m+n}$.

(2) $\dfrac{a^m}{a^n} = a^{m-n}$.

(3) $(a^m)^n = a^{mn}$.

(4) $(ab)^n = a^n b^n$.

(5) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

(6) $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$.

(7) $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$.

Simplifying expressions with exponents

Example

Simplify:

(1) $(2a^3b^2)(3ab^4)^3$.

(2) $\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4$.

(3) $\frac{6st^{-4}}{2s^{-2}t^2}$.

(4) $\left(\frac{y}{3z^3}\right)^{-2}$.

Simplifying expressions with exponents

Solution.

(1) $(2a^3b^2)(3ab^4)^3 = 54a^6b^{14}$.

(2) $\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4 = \frac{x^7y^5}{z^4}$.

(3) $\frac{6st^{-4}}{2s^{-2}t^2} = \frac{3s^3}{t^6}$.

(4) $\left(\frac{y}{3z^3}\right)^{-2} = \frac{9z^6}{y^2}$.

Scientific notation

Definition

Any $x \in \mathbb{R}^+$ may be written in scientific notation as

$$x = a \times 10^n, \quad 1 \leq a < 10 \text{ and } n \in \mathbb{Z}.$$

Examples of scientific notation

Scientific notation is extremely useful when denoting very large and very small numbers:

Example

- The distance between the earth and the star Proxima Centauri is approximately 3.99×10^{13} km:

$$3.99 \times 10^{13} = 39,900,000,000,000.$$

- The mass of a hydrogen atom is 1.66×10^{-24} g:

$$1.66 \times 10^{-24} = 0.00000000000000000000000166.$$

- $56,920 = 5.692 \times 10^{4}$.
- $0.000093 = 9.3 \times 10^{-5}$.

Radicals

Definition

- For an odd positive integer n, the principle nth root $b = \sqrt[n]{a}$ of $a \in \mathbb{R}$ is a real number such that $b^n = a$.
- For an even positive integer n, the principle nth root $b = \sqrt[n]{a}$ of $a \in \mathbb{R}^+$ is a positive real number such that $b^n = a$.
- For any $n \in \mathbb{Z}^+$, the principle nth root of 0 is 0.

Properties of radicals

Proposition

(1) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

(2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

(3) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

(4) $\sqrt[n]{a^n} = a$ *if n is odd.*

(5) $\sqrt[n]{a^n} = |a|$ *if n is even.*

Simplifying expressions with radicals

Example

Simplify:

(1) $\sqrt[3]{x^4}$.

(2) $\sqrt[4]{81x^8y^4}$.

(3) $\sqrt{32} + \sqrt{200}$.

(4) $\sqrt{25b} - \sqrt{b^3}$.

(5) $\sqrt{49x^2 + 49}$.

Simplifying expressions with radicals

Solution.

(1) $\sqrt[3]{x^4} = x\sqrt[3]{x}$.

(2) $\sqrt[4]{81x^8y^4} = 3x^2|y|$.

(3) $\sqrt{32} + \sqrt{200} = 14\sqrt{2}$.

(4) $\sqrt{25b} - \sqrt{b^3} = (5 - b)\sqrt{b}$.

(5) $\sqrt{49x^2 + 49} = 7\sqrt{x^2 + 1}$.

Rational exponents

Definition

For any $\frac{m}{n} \in \mathbb{Q}$ in lowest terms (i.e., irreducible fractions) with $n > 0$, define

$$a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

where $a \geq 0$ is required whenever n is even.

In particular, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Simplifying expressions with rational exponents

Example

(1) $(2a^3b^4)^{\frac{3}{2}}$.

(2) $\left(\frac{2x^{\frac{3}{4}}}{y^{\frac{1}{3}}}\right)^3\left(\frac{y^4}{x^{-\frac{1}{2}}}\right)$.

Simplifying expressions with rational exponents

Solution.

(1) $(2a^3b^4)^{\frac{3}{2}} = 2\sqrt{2}a^{\frac{9}{2}}b^6$.

(2) $\left(\frac{2x^{\frac{3}{4}}}{y^{\frac{1}{3}}}\right)^3\left(\frac{y^4}{x^{-\frac{1}{2}}}\right) = 8x^{\frac{11}{4}}y^3$.

Rationalizing the denominator

It is often useful to eliminate the radical in a denominator by rationalizing the denominator.
A fractional expression whose denominator contains no radicals is said to be in standard form.

Example

- $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.
- $\frac{1}{\sqrt[3]{5}} = \frac{\sqrt[3]{25}}{5}$.
- $\sqrt[7]{\frac{1}{a^2}} = \frac{\sqrt[7]{a^5}}{a}$.

Outline

Algebraic expressions

An algebraic expression is an expression that consists of

- constants (usually denoted by a, b, c),
- variables (usually denoted by x, y, z), and
- the algebraic operations (addition, subtraction, multiplication, division and exponentiation by a rational exponent).

Here are some examples:

- $2x^2 + 3x + c$,
- $\sqrt{x} + 10$,
- $\dfrac{y - 2z}{y^2 + 4}$.

Polynomials

Definition

A polynomial in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$. If $a_n \neq 0$, then n is called the degree of this polynomial. Each $a_k x^k$ that makes up the polynomial is called a term of the polynomial, and a_k is called the coefficient of x^k.

In particular, a polynomial with only one term is called a monomial, and by a binomial and a trinomial we mean a polynomial with two and three terms, respectively.

Operations of polynomials

Denote the set of all polynomials in the variable x with real coefficients as

$$\mathbb{R}[x],$$

then $\mathbb{R}[x]$ is closed with respect to addition, substraction and multiplication, which is in fact a ring in the terminology of modern algebra. This means that the multiplication of polynomials is distributive over addition:

$$f(g+h) = fg + fh,$$
$$(f+g)h = fh + gh.$$

Products of polynomials

Example

Find the product of polynomials

$$(2x + 3)(x^2 - 5x + 4).$$

Products of polynomials

Solution.

$$\begin{aligned}
&(2x+3)(x^2-5x+4)\\
=\ &2x(x^2-5x+4)+3(x^2-5x+4)\\
=\ &2x^3-7x^2-7x+12.
\end{aligned}$$

Products of polynomials

Proposition

(1) $(x+y)(x-y) = x^2 - y^2$.

(2) $(x+y)^2 = x^2 + 2xy + y^2$.

(2') $(x-y)^2 = x^2 - 2xy + y^2$.

(3) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

(3') $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.

Products of polynomials

Example

Find the products:

(1) $(3x+5)^2$.

(2) $(x^2-2)^3$.

Products of polynomials

Solution.

(1) $(3x+5)^2 = 9x^2 + 30x + 25.$

(2) $(x^2-2)^3 = x^6 - 6x^4 + 12x^2 - 8.$

□

Factoring polynomials

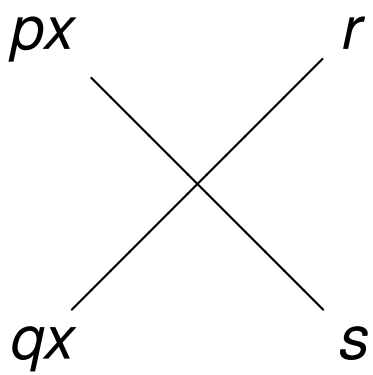
Just now we expand the products of polynomials. Sometimes we also need to reverse this process by factoring a polynomial as products of simpler ones.

$$(3x+5)^2 \xrightarrow{\text{Expanding}} 9x^2+30x+25$$

$$(3x+5)^2 \xleftarrow{\text{Factoring}} 9x^2+30x+25$$

Factoring trinomials

The basic technique to factor a trinomial of the form $ax^2 + bx + c$ is the Criss-Cross method. Explicitly, we find factors p, q of a, factors r, s of c by trial and error such that $pq = a$, $rs = c$ and $ps + qr = b$:



Then $ax^2 + bx + c = (px + r)(qx + s)$.

Factoring trinomials

Example

Factor each expression:

(1) $x^2 - 2x - 3$.

(2) $10x^2 - 11x - 6$.

Factoring trinomials

Solution.

(1) $x^2 - 2x - 3 = (x - 3)(x + 1)$.

(2) $10x^2 - 11x - 6 = (5x + 2)(2x - 3)$.

□

Factoring formulas

Proposition

(1) $x^2 - y^2 = (x + y)(x - y)$.

(2) $x^2 + 2xy + y^2 = (x + y)^2$.

(2') $x^2 - 2xy + y^2 = (x - y)^2$.

(3) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(3') $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Factoring polynomials

Example

Factor each expression:

(1) $x^2 + 6x + 9$.

(2) $27x^3 - 1$.

(3) $2x^4 - 8x^2$.

(4) $x^3 + x^2 + 4x + 4$.

(5) $x^3 - 2x^2 - 9x + 18$.

Factoring polynomials

Solution.

(1) $x^2 + 6x + 9 = (x + 3)^2$.

(2) $27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$.

(3) $2x^4 - 8x^2 = 2x^2(x^2 - 4) = 2x^2(x - 2)(x + 2)$.

(4) $x^3 + x^2 + 4x + 4 = (x + 1)(x^2 + 4)$.

(5) $x^3 - 2x^2 - 9x + 18 = (x - 2)(x^2 - 9) = (x - 2)(x - 3)(x + 3)$.

□