Fundamentals of Mathematics (MATH 1510)

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September 21-25, 2015

Outline

1. Rational Expressions
2. Equations

Rational expressions

A quotient of two algebraic expressions is called a fractional expression.

A rational expression is a fractional expression in which both the numerator and the denominator are polynomials.

Example

$\dfrac{2x}{x-1}$, $\dfrac{y-2}{y^2+4}$, $\dfrac{x^3-x}{x^2-5x+6}$ are rational expressions.

$\dfrac{x}{\sqrt{x^2+1}}$ is a fractional expression but not a rational expression.

The domain of an algebraic expression

The domain of an algebraic expression is the set of real numbers that the variable is permitted to have. So far we have encountered two restrictions for the variable:

Expression	Domain
$\frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$
$\sqrt[2k]{x}$, where $k \in \mathbb{Z}^+$	$[0, \infty)$

The domain of an algebraic expression

Example

Find the domains of the following expressions:

(1) $2x^2 + 3x - 1$.

(2) $\dfrac{x}{x^2 - 5x + 6}$.

(3) $\dfrac{\sqrt{x}}{x - 5}$.

The domain of an algebraic expression

Solution.

(1) $\mathbb{R}$, or $(-\infty, \infty)$.

(2) $\{x \mid x \neq 2 \text{ and } x \neq 3\}$, or $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

(3) $\{x \mid x \geq 0 \text{ and } x \neq 5\}$, or $[0, 5) \cup (5, \infty)$.

$\square$

Simplifying rational expressions

Example

It is possible to simplify a rational expression by cancelling common factors from the numerator and denominator. For example:

$$\frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{x+1}{x+2}.$$

One should be aware that $\frac{x^2-1}{x^2+x-2}$ and $\frac{x+1}{x+2}$ have different domains: $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$ for the former, and $(-\infty,-2)\cup(-2,\infty)$ for the latter, although they are equal in the intersection of their domains.

Simplifying algebraic expressions

Example

Simplify the following expressions and specify the domain of the variable(s):

(1) $\dfrac{x^2+2x-3}{x^2+8x+16}\cdot\dfrac{3x+12}{x-1}$.

(2) $\dfrac{x-4}{x^2-4}\div\dfrac{x^2-3x-4}{x^2+5x+6}$.

(3) $\dfrac{3}{x-1}+\dfrac{x}{x+2}$.

(4) $\dfrac{1}{x^2-1}-\dfrac{2}{(x+1)^2}$.

(5) $\dfrac{\frac{x}{y}+1}{1-\frac{y}{x}}$.

Simplifying algebraic expressions

Solution.

(1) For $x \in (-\infty, -4) \cup (-4, 1) \cup (1, \infty)$,

$$\begin{aligned}
&\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} \\
= &\frac{(x-1)(x+3)}{(x+4)^2} \cdot \frac{3(x+4)}{x-1} \\
= &\frac{3(x+3)}{x+4}.
\end{aligned}$$

Simplifying algebraic expressions

(2) For $x \in \{x \in \mathbb{R} \mid x \neq -3, -2, -1, 2 \text{ or } 4\}$,

$$\begin{aligned}
&\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} \\
=& \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \\
=& \frac{x-4}{(x-2)(x+2)} \cdot \frac{(x+2)(x+3)}{(x-4)(x+1)} \\
=& \frac{x+3}{(x-2)(x+1)}.
\end{aligned}$$

Simplifying algebraic expressions

(3) For $x \in (-\infty, -2) \cup (-2, 1) \cup (1, \infty)$,

$$\begin{aligned} & \frac{3}{x-1} + \frac{x}{x+2} \\ = & \frac{3(x+2) + x(x-1)}{(x-1)(x+2)} \\ = & \frac{x^2 + 2x + 6}{(x-1)(x+2)}. \end{aligned}$$

Simplifying algebraic expressions

(4) For $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$,

$$\begin{aligned} & \frac{1}{x^2 - 1} - \frac{2}{(x+1)^2} \\ = & \frac{(x+1) - 2(x-1)}{(x-1)(x+1)^2} \\ = & \frac{3 - x}{(x-1)(x+1)^2}. \end{aligned}$$

Simplifying algebraic expressions

(5) For $(x, y) \in \{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0 \text{ and } x \neq y\}$,

$$\frac{\frac{x}{y} + 1}{1 - \frac{y}{x}} = \frac{x^2 + xy}{xy - y^2} = \frac{x(x + y)}{y(x - y)}.$$

□

Outline

1. Rational Expressions

2. Equations

Equations

A equation is a statement that two mathematical expressions are equal. Most equations we study contain variables.

The values of the unknown variable(s) that make the equation true are called the solutions or roots of the equation, and the process of finding the solutions is called solving the equation.

Two equations with exactly the same solutions are called equivalent equations. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the “equal” sign.

Equations

Proposition

For algebraic expressions A, B, C:

(1) $A = B \iff A + C = B + C$;

(2) $A = B \iff CA = CB \; (C \neq 0)$.

Linear equations

Definition

A linear equation in one variable is an equation equivalent to one of the form

$$ax + b = 0,$$

where $a, b \in \mathbb{R}$ are constants and x is the variable.

Linear equations

For example,

$$4x - 5 = 3,$$
$$2x = \frac{1}{2}x - 7,$$
$$x - 6 = \frac{x}{3}$$

are linear equations, but

$$x^2 + 2x = 8,$$
$$\sqrt{x} - 6x = 0,$$
$$\frac{3}{x} - 2x = 1$$

are nonlinear equations.

Solving linear equations

Example

Solve the equation

$$7x - 4 = 3x + 8.$$

Solving linear equations

Solution.

$$7x - 4 = 3x + 8,$$
$$4x = 12,$$
$$x = 3.$$

□

Quadratic equations

Definition

A quadratic equation in one variable is an equation equivalent to one of the form

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{R}$ are constants with $a \neq 0$ and x is the variable.

Solving quadratic equations

Factoring is an important method of solving quadratic equations as the following proposition reveals:

Proposition (Zero-product property)

$AB = 0$ if, and only if, $A = 0$ or $B = 0$.

Solving quadratic equations

Example

Solve the equation

$$x^2 + 5x = 24.$$

Solving quadratic equations

Solution.

$$
\begin{aligned}
x^2 + 5x &= 24, \\
x^2 + 5x - 24 &= 0, \\
(x+8)(x-3) &= 0, \\
x = -8 \text{ or } x &= 3.
\end{aligned}
$$

The solutions are $x = -8$ and $x = 3$. □

Solving quadratic equations

Proposition

The solutions of the equation $x^2 = c$ *are* $x = \sqrt{c}$ *and* $x = -\sqrt{c}$.

Solving quadratic equations

Example

Solve the equations:

(1) $x^2 = 5.$

(2) $(x-4)^2 = 5.$

Solving quadratic equations

Solution.

(1) The solutions are $x = \pm\sqrt{5}$.

(2)

$$\begin{aligned}(x-4)^2 &= 5,\\ x-4 &= \pm\sqrt{5},\\ x &= 4\pm\sqrt{5}.\end{aligned}$$

□

Solving quadratic equations

As we saw in the above example, a quadratic equation of the form $(x+a)^2 = c$ can be solved by taking the square root of each side.

For a quadratic equation that cannot be readily factorize, we may solve it by completing the square. Explicitly,

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}.$$

Solving quadratic equations

Example

Solve the following equations:

(1) $x^2 - 8x + 13 = 0.$

(2) $3x^2 - 12x + 6 = 0.$

Solving quadratic equations

Solution.

(1)

$$
\begin{aligned}
x^2 - 8x + 13 &= 0, \\
x^2 - 8x + 16 &= 3, \\
(x-4)^2 &= 3, \\
x - 4 &= \pm\sqrt{3}, \\
x &= 4 \pm \sqrt{3}.
\end{aligned}
$$

Solving quadratic equations

(2)

$$\begin{aligned} 3x^2 - 12x + 6 &= 0, \\ x^2 - 4x + 2 &= 0, \\ x^2 - 4x + 4 &= 2, \\ (x-2)^2 &= 2, \\ x - 2 &= \pm\sqrt{2}, \\ x &= 2 \pm \sqrt{2}. \end{aligned}$$

□

Solving quadratic equations

Proposition (The quadratic formula)

The roots of the quadratic equation $ax^2 + bx + c = 0$, *where* $a \neq 0$, *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solving quadratic equations

Definition

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is

$$D = b^2 - 4ac.$$

Solving quadratic equations

Proposition

For the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$*:*

(1) *The equation has two distinct real solutions if and only if* $D > 0$.

(2) *The equation has exactly one real solution if and only if* $D = 0$.

(3) *The equation has no real solution if and only if* $D < 0$.

Solving quadratic equations

Example

Find all real solutions of each equation:

(1) $3x^2 - 5x - 1 = 0.$

(2) $4x^2 + 12x + 9 = 0.$

(3) $\frac{1}{3}x^2 - 2x + 4 = 0.$

Solving quadratic equations

Solution.

(1) The discriminant $D = (-5)^2 - 4 \cdot 3(-1) = 37 > 0$. The equation has two real solutions

$$x = \frac{5 \pm \sqrt{37}}{6}.$$

(2) The discriminant $D = 12^2 - 4 \cdot 4 \cdot 9 = 0$. The equation has exactly one real solution

$$x = \frac{-12}{8} = -\frac{3}{2}.$$

(3) The discriminant $D = (-2)^2 - 4 \cdot \frac{1}{3} \cdot 4 = -\frac{4}{3} < 0$. The equation has no real solution. $\square$

Solving other types of equations

Example

Solve the equation

$$x^4 - 8x^2 + 8 = 0.$$

Solving other types of equations

Solution.

Using the quadratic formula one has

$$x^2 = \frac{8 \pm \sqrt{32}}{2} = 4 \pm 2\sqrt{2}.$$

Therefore

$$x = \pm\sqrt{4 \pm 2\sqrt{2}}.$$

□

Solving other types of equations

Example

Solve the equation

$$\frac{3}{x} - \frac{2}{x-3} = -\frac{12}{x^2-9}.$$

Solving other types of equations

Solution.

Multiply $x(x^2-9)$ on both sides one has

$$\begin{aligned} 3(x^2-9)-2x(x+3) &= -12x, \\ x^2+6x-27 &= 0, \\ (x+9)(x-3) &= 0. \end{aligned}$$

Thus $x=-9$ or $x=3$. But $x=3$ is an extraneous solution since 3 is outside the domain of the algebraic expression in the equation. Therefore the solution is

$$x=-9.$$

□

Solving other types of equations

Example

Solve the equation

$$2x = 1 - \sqrt{2 - x}.$$

Solving other types of equations

Solution.

$$
\begin{aligned}
2x - 1 &= -\sqrt{2 - x}, \\
(2x - 1)^2 &= 2 - x, \\
4x^2 - 3x - 1 &= 0, \\
(4x + 1)(x - 1) &= 0.
\end{aligned}
$$

Thus $x = -\frac{1}{4}$ or $x = 1$. But $x = 1$ is an extraneous solution since it does not satisfy the original equation. Therefore the solution is

$$x = -\frac{1}{4}.$$

□

Extraneous solutions

Question

In the above two examples, where do the extraneous solutions come from?

Extraneous solutions

Answer.

In the first example, we convert the equation

$$\frac{3}{x} - \frac{2}{x-3} = -\frac{12}{x^2-9} \tag{1}$$

to

$$3(x^2-9) - 2x(x+3) = -12x. \tag{2}$$

The domain of the algebraic expressions on both sides of Equation (1) are respectively

$$(-\infty, 0) \cup (0, 3) \cup (3, \infty) \quad \text{and} \quad (-\infty, -3) \cup (-3, 3) \cup (3, \infty),$$

but the domain of the algebraic expressions on both sides of Equation (2) are $\mathbb{R}$. Therefore, solutions of (2) may not be solutions of (1).

Extraneous solutions

In the second example, we convert the equation

$$2x - 1 = -\sqrt{2 - x} \tag{3}$$

to

$$(2x - 1)^2 = 2 - x \tag{4}$$

by taking the square on both sides. However, although (3) implies (4), the reverse implication may not be true. For example,

$$-3 \neq 3$$

but

$$9 = (-3)^2 = 3^2 = 9.$$

Therefore, solutions of (4) may not be solutions of (3). $\square$