# Fundamentals of Mathematics (MATH 1510)

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# Outline

1. Complex Numbers
2. Inequalities
3. Modeling

# Complex numbers



## Definition

A complex number is an expression of the form

$$a + bi,$$

where $a, b \in \mathbb{R}$ and $i^2 = -1$; $a$ is called the real part and $b$ is called the imaginary part. For two complex numbers $a + bi$, $c + di$,

$$a + bi = c + di \iff a = c \text{ and } b = d.$$

A complex number whose real part is zero is said to be purely imaginary.

# Complex numbers



If PJ stands for Poor Joke,
then P+iJ is a Complex Poor Joke,
and you did not laugh because the Joke part is imaginary.

# Arithmetic operations on complex numbers



**Proposition**

(1) $(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i.$

(2) $(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$

# Arithmetic operations on complex numbers



## Example

Calculate:

(1) $(3+5i)+(4-2i)$.

(2) $(3+5i)-(4-2i)$.

(3) $(3+5i)(4-2i)$.

(4) $i^{23}$.

# Arithmetic operations on complex numbers

**Solution.**

(1) $(3+5i)+(4-2i)=7+3i$.

(2) $(3+5i)-(4-2i)=-1+7i$.

(3) $(3+5i)(4-2i)=22+14i$.

(4) $i^{23}=-i$.

$\square$

# Complex conjugates



The complex conjugate of a complex number $z = a + bi$ is

$$\overline{z} = a - bi.$$

The product of a complex number and its conjugate is always a nonnegative real number:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

# Arithmetic operations on complex numbers



## Proposition

*The division of two complex numbers is calculated as:*

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

# Arithmetic operations on complex numbers



## Example

Calculate:

(1) $\dfrac{3+5i}{1-2i}$.

(2) $\dfrac{7+3i}{4i}$.

# Arithmetic operations on complex numbers



**Solution.**

(1) $$\frac{3+5i}{1-2i} = \frac{(3+5i)(1+2i)}{(1-2i)(1+2i)} = -\frac{7}{5} + \frac{11}{5}i.$$

(2) $$\frac{7+3i}{4i} = \frac{(7+3i)i}{4i \cdot i} = \frac{3}{4} - \frac{7}{4}i.$$

□

# Square roots of negative numbers



**Definition**

For any $r \in \mathbb{R}^+$, the principle square root of $-r$ is

$$\sqrt{-r} = i\sqrt{r}.$$

The two square roots of $-r$ are $\pm i\sqrt{r}$.

We usually write $i\sqrt{r}$ instead of $\sqrt{r}i$ to avoid confusion with $\sqrt{ri}$.

# Square roots of negative numbers



Although

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

when $a, b \in \mathbb{R}^+$, this is not true when $a, b \in \mathbb{R}^-$. For example:

$$\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = -\sqrt{6},$$

$$\sqrt{(-2)(-3)} = \sqrt{6}.$$

# Square roots of negative numbers



**Example**

Calculate $(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$.

# Square roots of negative numbers



**Solution.**

$$
\begin{aligned}
&(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4}) \\
&= (2\sqrt{3} - i\sqrt{3})(3 + 2i) \\
&= (6\sqrt{3} + 2\sqrt{3}) + i(4\sqrt{3} - 3\sqrt{3}) \\
&= 8\sqrt{3} + i\sqrt{3}.
\end{aligned}
$$

□

# Complex solutions of quadratic equations



We already know that the solutions of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, the equation has no real solution.

However, in the complex number system, the equation always have solutions even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \quad (\text{if } b^2 - 4ac < 0).$$

# Complex solutions of quadratic equations



## Example

Solve the following equations:

(1) $x^2 + 9 = 0.$

(2) $x^2 + 4x + 5 = 0.$

(3) $\frac{1}{3}x^2 - 2x + 4 = 0.$

# Complex solutions of quadratic equations



**Solution.**

(1) $x = \pm 3i$.

(2) $$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm i.$$

# Complex solutions of quadratic equations



(3)

$$\begin{aligned} \frac{1}{3}x^2 - 2x + 4 &= 0, \\ x^2 - 6x + 12 &= 0, \\ (x-3)^2 &= -3, \\ x - 3 &= \pm i\sqrt{3}, \\ x &= 3 \pm i\sqrt{3}. \end{aligned}$$

□

# Complex conjugate root theorem



**Theorem**

*For any equation equivalent to the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

*where* $a_0, a_1, \ldots, a_n \in \mathbb{R}$*, if* $c + di$ $(c, d \in \mathbb{R})$ *is a solution of this equation, then so is its conjugate* $c - di$*.*

# Outline

# Inequalities



Inequalities look like equations but the equal sign is replaced by $<$, $>$, $\leq$ or $\geq$. For example:

$$4x + 7 \leq 19.$$

To solve an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions which form the solution set of the inequality.

# Rules for inequalities



**Proposition**

(1) $A \leq B \iff A + C \leq B + C$.

(2) $A \leq B \iff A - C \leq B - C$.

(3) *If* $C > 0$, *then* $A \leq B \iff CA \leq CB$.

(4) *If* $C < 0$, *then* $A \leq B \iff CA \geq CB$.

(5) *If* $A > 0$ *and* $B > 0$, *then* $A \leq B \iff \dfrac{1}{A} \geq \dfrac{1}{B}$.

(6) *If* $A \leq B$ *and* $C \leq D$, *then* $A + C \leq B + D$.

(7) *If* $A \leq B$ *and* $B \leq C$, *then* $A \leq C$.

# Solving linear inequalities

## Example

Solve the inequality

$$3x < 9x + 4.$$

# Solving linear inequalities



**Solution.**

$$\begin{aligned} 3x &< 9x + 4, \\ -6x &< 4, \\ 6x &> -4, \\ x &> -\frac{2}{3}. \end{aligned}$$

So the solution set is $\left(-\frac{2}{3}, \infty\right)$. □

# Solving nonlinear inequalities



**Example**

Solve the following inequalities:

(1) $x^2 \leq 5x - 6$.

(2) $x(x-1)^2(x-3) < 0$.

(3) $\dfrac{1+x}{1-x} \geq 1$.

# Solving nonlinear inequalities



**Solution.**

(1)

$$\begin{aligned} x^2 &\leq 5x - 6, \\ x^2 - 5x + 6 &\leq 0, \\ (x-2)(x-3) &\leq 0. \end{aligned}$$

Checking the sign of $x - 2$, $x - 3$ in the intervals

$$(-\infty, 2), (2, 3), (3, \infty),$$

we obtain the solution set $[2, 3]$.

# Solving nonlinear inequalities



(2) Checking the sign of $x$, $(x-1)^2$ and $x-3$ in the intervals

$$(-\infty, 0), (0, 1), (1, 3), (3, \infty),$$

we obtain the solution set

$$(0, 1) \cup (1, 3).$$

# Solving nonlinear inequalities



(3)

$$\frac{1+x}{1-x} \geq 1,$$
$$\frac{1+x}{1-x} - 1 \geq 0,$$
$$\frac{2x}{1-x} \geq 0.$$

Checking the sign of $2x$, $1-x$ in the intervals

$$(-\infty, 0), (0, 1), (1, \infty),$$

we obtain the solution set $[0, 1)$.

□

# Absolute value inequalities



**Proposition**

*For any* $c \in \mathbb{R}^+$*:*

(1) $|x| \leq c \iff -c \leq x \leq c.$

(2) $|x| \geq c \iff x \leq -c \text{ or } x \geq c.$

# Solving absolute value inequalities

## Example

Solve the following inequalities:

(1) $|x - 5| < 2$.

(2) $|3x + 2| \geq 4$.

# Solving absolute value inequalities



**Solution.**

(1)

$$\begin{aligned} |x-5| &< 2, \\ -2 < x-5 &< 2, \\ 3 < x &< 7. \end{aligned}$$

So the solution set is $(3, 7)$.

# Solving absolute value inequalities



(2)

$$
\begin{aligned}
|3x+2| &\geq 4, \\
3x+2 \geq 4 \quad &\text{or} \quad 3x+2 \leq -4, \\
3x \geq 2 \quad &\text{or} \quad 3x \leq -6, \\
x \geq \frac{2}{3} \quad &\text{or} \quad x \leq -2,
\end{aligned}
$$

So the solution set is $(-\infty, -2] \cup \left[\frac{2}{3}, \infty\right)$.

□

# Outline

# Modeling with equations



**Example**

A car rental company charges 30 dollars a day and 15 cents a mile for renting a car. Helen rents a car for two days, and her bill comes to 108 dollars. How many miles did she drive?

# Modeling with equations

**Solution.**

Let $x$ be the number of miles driven. Then

$$\begin{aligned} 30 \cdot 2 + 0.15x &= 108, \\ 0.15x &= 48, \\ x &= 320. \end{aligned}$$

So, Helen drove her rental car 320 miles. □

# Modeling with equations

**Example**

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 $ft^2$. Find the dimensions of the lot.

# Modeling with equations



**Solution.**

Let $x$ be the width of the lot. Then

$$\begin{aligned} x(x+8) &= 2900, \\ x^2+8x-2900 &= 0, \\ (x-50)(x+58) &= 0. \end{aligned}$$

Since the width of the lot must be a positive number, we conclude that $x = 50$ ft.

So, the width and the length of the lot are respectively 50 ft and 58 ft. □

# Modeling with inequalities



**Example**

A carnival has two plans for tickets:

- Plan A: 5 dollars as entrance fee and 25 cents for each ride.
- Plan B: 2 dollars as entrance fee and 50 cents for each ride.

How many rides would you have to take for Plan A to be less expensive than Plan B?

# Modeling with inequalities



**Solution.**

Let $x$ be the number of rides. Then

$$\begin{aligned} 5 + 0.25x &< 2 + 0.50x, \\ 0.25x &> 3, \\ x &> 12. \end{aligned}$$

So if you plan to take more than 12 rides, Plan A is less expensive. □