Fundamentals of Mathematics (MATH 1510)

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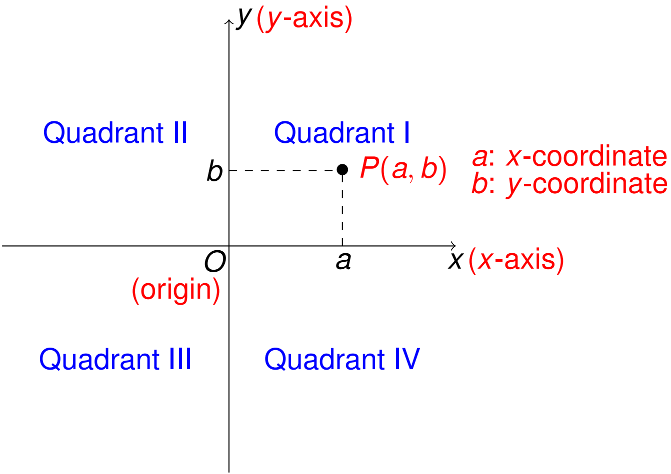
Outline

The coordinate plane

The coordinate plane or Cartesian plane:



Graphing regions in the coordinate plane

Example

Sketch the following sets in the coordinate plane:

(1) $\{(x, y) \mid x \geq 0\}$.

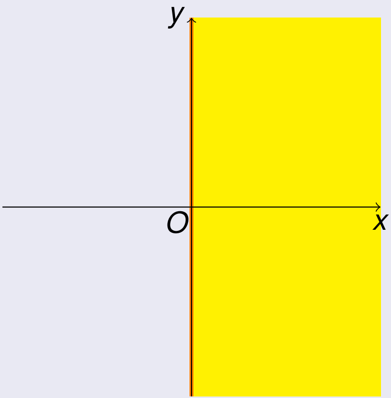
(2) $\{(x, y) \mid y = 1\}$.

(3) $\{(x, y) \mid |y| < 1\}$.

Graphing regions in the coordinate plane

Solution.

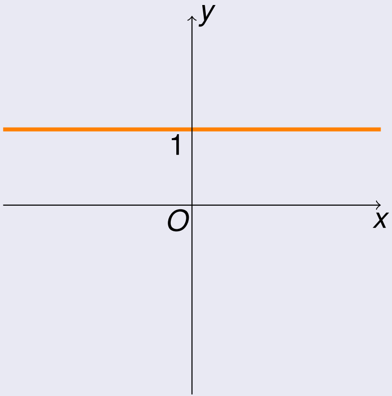
(1)



Graphing regions in the coordinate plane

Solution.

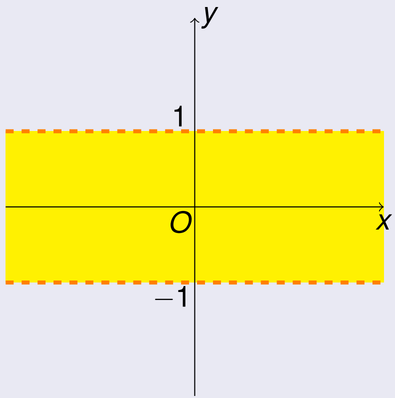
(2)



Graphing regions in the coordinate plane

Solution.

(3)



□

The distance formula

Theorem

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This theorem can be easily proved by Pythagorean Theorem.

Applying the distance formula

Example

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

Applying the distance formula

Solution.

Since

$$d(P,A) = \sqrt{(1-5)^2 + (-2-3)^2} = \sqrt{41},$$
$$d(Q,A) = \sqrt{(8-5)^2 + (9-3)^2} = \sqrt{45},$$

it follows that P is closer to A. □

The midpoint formula

Theorem

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Applying the midpoint formula

Example

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

Applying the midpoint formula

Solution.

The midpoint of the diagonal *PR* is

$$\left(\frac{1+5}{2}, \frac{2+9}{2}\right) = \left(3, \frac{11}{2}\right),$$

and the midpoint of the diagonal *QS* is

$$\left(\frac{4+2}{2}, \frac{4+7}{2}\right) = \left(3, \frac{11}{2}\right).$$

So the two diagonals of the quadrilateral *PQRS* bisect the other, and thus it is a parallelogram. □

Graphs of equations in two variables

An equation in two variables, such as $y = x^2 + 1$, expresses a relationship between two quantities.

The graph of an equation in x and y is the set of points (x, y) in the coordinate plane that satisfy the equation.

Intercepts

The x-coordinates (resp. y-coordinates) of the points where a graph intersects the x-axis (resp. y-axis) are called the x-intercepts (resp. y-intercepts) of the graph.

x-intercepts (resp. y-coordinates) can be found by setting $y = 0$ (resp. $x = 0$) and solve for x (resp. y).

Finding intercepts

Example

Find the x-intercepts and y-intercepts of the graph of the equation $y = x^2 - 2$.

Finding intercepts

Solution.

Let $y = 0$ one has $x^2 - 2 = 0$. Solving for x one obtains the x-intercepts $\sqrt{2}$ and $-\sqrt{2}$.

Next, we set $x = 0$ and get $y = -2$. Thus the y-intercept is -2. □

Circles

Definition

An equation of the circle with center (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2.$$

This is called the standard form for the equation of the circle. If the center of the circle is $(0, 0)$, then the equation is

$$x^2+y^2=r^2.$$

Finding an equation of a circle

Example

Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

Finding an equation of a circle

Solution.

The center of the circle is the midpoint of PQ:

$$\left(\frac{1+5}{2}, \frac{8-6}{2}\right) = (3, 1).$$

Let the radius of the circle be r. Then

$$r^2 = (1-3)^2 + (8-1)^2 = 53.$$

Therefore the equation of the circle is

$$(x-3)^2 + (y-1)^2 = 53.$$

□

Symmetry

A graph is symmetric w.r.t. the y-axis if whenever the point (x, y) is on the graph, then so is $(-x, y)$.

A graph is symmetric w.r.t. the x-axis if whenever the point (x, y) is on the graph, then so is $(x, -y)$.

A graph is symmetric w.r.t. the origin if whenever the point (x, y) is on the graph, then so is $(-x, -y)$.

Symmetry

Example

- The graph of $y = x^2$ is symmetric w.r.t. the y-axis.
- The graph of $x = y^2$ is symmetric w.r.t. the x-axis.
- The graph of $y = x$ is symmetric w.r.t. the origin.

Outline

The slope of a line

If a line lies in a coordinate plane, then the run is the change in the x-coordinate and the rise is the corresponding change in the y-coordinate between any two points on the line.

The slope of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

The slope of a line

Definition

The slope m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of a vertical line is not defined.

The slope of a line

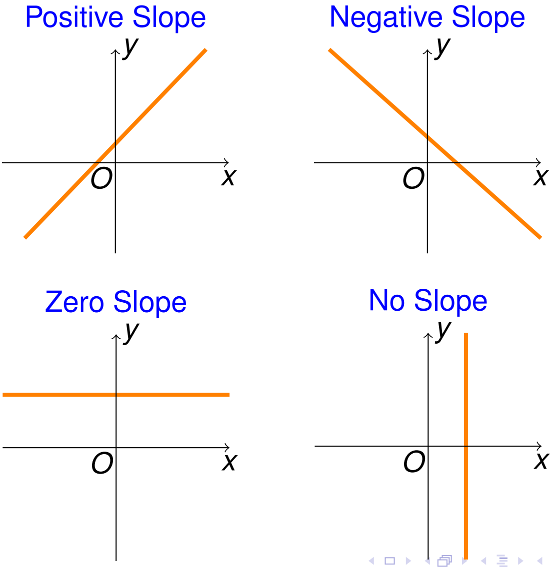
The slope is independent of which two points are chosen on the line. That is, if

$$(x_1, y_1), (x_2, y_2), (x_1', y_1'), (x_2', y_2')$$

belong to the same nonvertical line, then it always holds that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}.$$

The slope of a line



Point-slope form of the equation of a line

Definition

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1).$$

Point-slope form of the equation of a line

Example

The equation of the line through $(1, -3)$ with slope $-\dfrac{1}{2}$ is

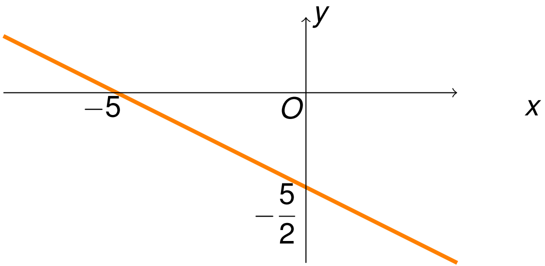
$$y + 3 = -\frac{1}{2}(x - 1),$$

i.e.,

$$x + 2y + 5 = 0.$$

Point-slope form of the equation of a line

To sketch the graph of $x + 2y + 5 = 0$, note that its x-intercept is -5, and y-intercept is $-\dfrac{5}{2}$. Thus the line passes through $(-5, 0)$ and $\left(0, -\dfrac{5}{2}\right)$:



Slope-intercept form of the equation of a line

Definition

An equation of the line that has slope m and y-intercept b is

$$y = mx + b.$$

Slope-intercept form of the equation of a line

Example

(1) Find an equation of the line with slope 3 and y-intercept -2.

(2) Find the slope and y-intercept of the line $3y - 2x = 1$.

Slope-intercept form of the equation of a line

Solution.

(1) $y = 3x - 2$.

(2) Rewrite the equation $3y - 2x = 1$ as

$$y = \frac{2}{3}x + \frac{1}{3},$$

one soon has the slope $\frac{2}{3}$ and the y-intercept $\frac{1}{3}$.

□

Vertical and horizontal lines

Definition

- An equation of the vertical line through (a, b) is $x = a$.
- An equation of the horizontal line through (a, b) is $y = b$.

General equation of a line

A linear equation in the variables x and y is an equation of the form

$$Ax + By + C = 0,$$

where A, B, C are constants and A, B are not both 0.

General equation of a line

Theorem

The graph of every linear equation

$$Ax + By + C = 0 \quad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

General equation of a line

Example

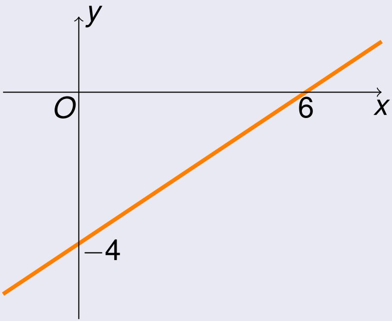
Sketch the graph of the linear equation

$$2x - 3y - 12 = 0.$$

General equation of a line

Solution.

Note that its x-intercept is 6, and y-intercept is -4. Thus the line passes through $(6, 0)$ and $(0, -4)$:



□

Parallel lines

Definition

Two nonvertical lines are parallel if and only if they have the same slope.

Parallel lines

Example

Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Parallel lines

Solution.

Writing the linear equation $4x + 6y + 5 = 0$ in the slope-intercept form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

one immediately has its slope $-\frac{2}{3}$. Thus the required line has the equation

$$y - 2 = -\frac{2}{3}(x - 5),$$

i.e.,

$$2x + 3y - 16 = 0.$$

□

Perpendicular lines

Definition

Two lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 m_2 = -1.$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

Perpendicular lines

Example

Show that the points $P(3, 3)$, $Q(8, 17)$ and $R(11, 5)$ are the vertices of a right triangle.

Perpendicular lines

Solution.

The slopes of the lines containing *PR* and *QR* are, respectively,

$$m_1 = \frac{5-3}{11-3} = \frac{1}{4} \quad \text{and} \quad m_2 = \frac{5-17}{11-8} = -4.$$

Since $m_1 m_2 = -1$, *PR* is perpendicular to *QR*, and consequently, *PQR* is a right triangle. □