# Fundamentals of Mathematics (MATH 1510)

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# Outline



1. Functions
2. Graphs of Functions

# Functions



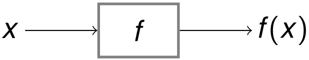
**Definition**

A function is a rule that assigns to each element $x$ in a set $A$ exactly one element, called $f(x)$, in a set $B$.

# Functions



Intuitively, a function may be thought of as a program. Given an input $x$, there will be a unique output $f(x)$. The same output may be produced by more than one input, but each input gives only one output.



# Functions



Given a function $f : A \longrightarrow B$:

- $A$ is called the domain of $f$.
- $B$ is called the codomain of $f$.
- The range of $f$ is a subset of $B$ given by

$$\{f(x) \mid x \in A\}.$$

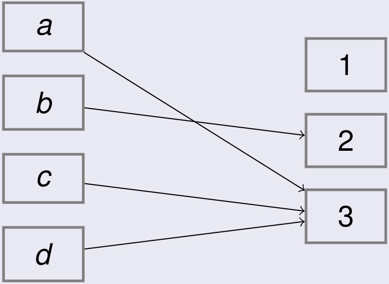
- If $f(a) = b$, then
  - $b$ is called the image of $a$ under $f$,
  - $a$ is called a preimage (or an inverse image) of $b$ under $f$.
- If we write $y = f(x)$, then $x$ is called a independent variable and $y$ is called a dependent variable.

# Examples of functions

## Example

Consider a function $f : A \longrightarrow B$ with the following assignments:

$$A = \{a, b, c, d\} \qquad B = \{1, 2, 3\}$$



# Examples of functions

Determine:

(1) $f(a)$.
(2) The image of $d$.
(3) The domain of $f$.
(4) The codomain of $f$.
(5) The preimage(s) of 1.
(6) The preimage(s) of 3.
(7) The range of $f$.

# Examples of functions



**Solution.**

(1) $f(a) = 3$.

(2) The image of $d$ is 3.

(3) The domain of $f$ is $A = \{a, b, c, d\}$.

(4) The codomain of $f$ is $B = \{1, 2, 3\}$.

(5) 1 has no preimage.

(6) The preimages of 3 are $a, c, d$.

(7) The range of $f$ is $\{2, 3\}$.

$\square$

# The domain of a function

The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2 \quad (0 \leq x \leq 5),$$

then the domain of $f$ is $[0, 5]$.

If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention the domain of the function is the domain of the algebraic expression; that is, the set of all real numbers for which the expression is defined.

# The domain of a function



## Example

Find the domain of each function:

(1) $f(x) = \dfrac{1}{x^2 - x}$.

(2) $g(x) = \sqrt{9 - x^2}$.

(3) $h(t) = \dfrac{t}{\sqrt{t+1}}$.

# The domain of a function



**Solution.**

(1) $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ or $\{x \mid x \neq 0 \text{ and } x \neq 1\}$.

(2) $[-3, 3]$.

(3) $(-1, \infty)$.

□

# Outline

# The graph of a function



**Definition**

If $f$ is a function with domain $A$, then the graph of $f$ is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}.$$

In other words, the graph of $f$ is the set of all points $(x, y)$ such that $y = f(x)$; that is, the graph of $f$ is the graph of the equation $y = f(x)$.

# Linear functions



A function $f$ of the form

$$f(x) = mx + b$$

is called a linear function because its graph is the graph of the linear equation $y = mx + b$.

A special case of a linear function occurs when the slope $m = 0$ and the function reduces to

$$f(x) = b,$$

where $b$ is a given number. Such $f$ is called a constant function because all its values are the same number, namely, $b$. Its graph is the horizontal line $y = b$.

# Power functions and root functions



Functions of the form

$$f(x) = x^n$$

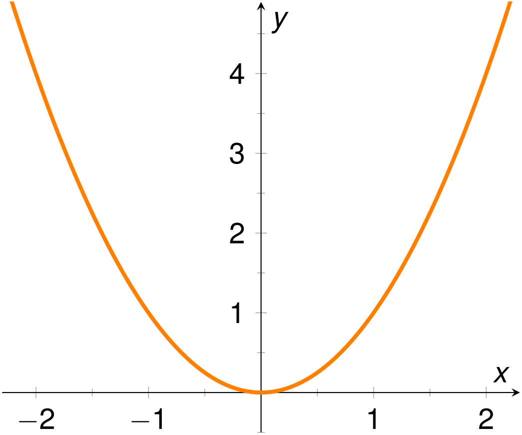
are called power functions, and functions of the form

$$f(x) = x^{\frac{1}{n}}$$

are called root functions.

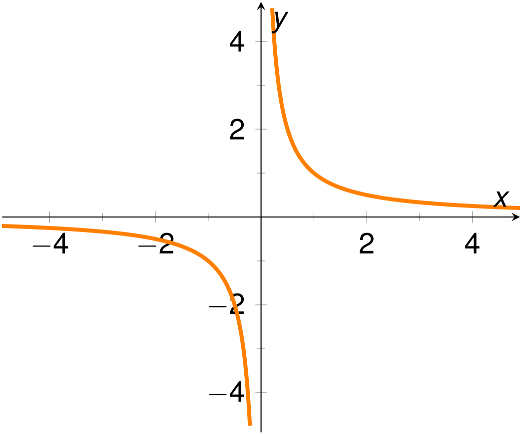
# Power functions and root functions

The graph of $f(x) = x^2$ is a parabola:



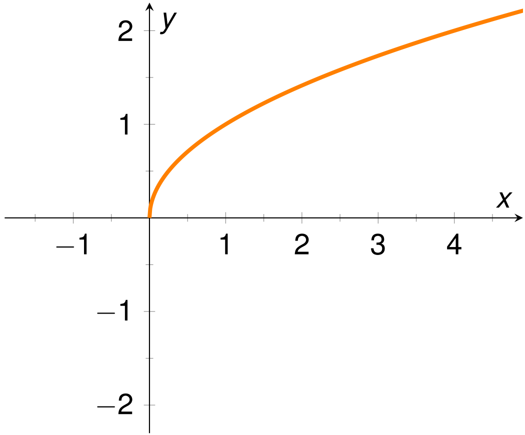
# Power functions and root functions

The graph of the reciprocal function $f(x) = \dfrac{1}{x}$ is a hyperbola:



# Power functions and root functions

The graph of the square root function $f(x) = \sqrt{x}$:



# Graph of a piecewise defined function



## Example

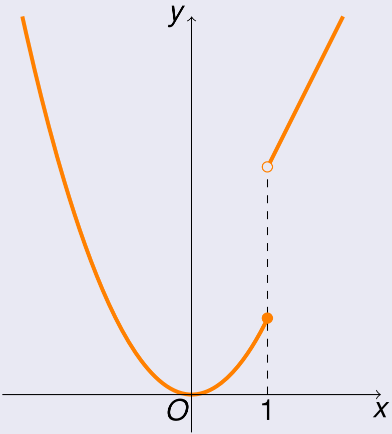
Sketch the graph of the function

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1, \\ 2x + 1, & \text{if } x > 1. \end{cases}$$

# Graph of a piecewise defined function



## Solution.

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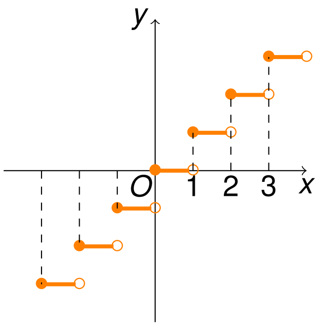
# Graph of a piecewise defined function

The greatest integer function is defined by

$$[[x]] = \text{greatest integer less than or equal to } x.$$

For example,

$$[[2.3]] = 2,\ [[2]] = 2,\ [[-0.5]] = -1.$$

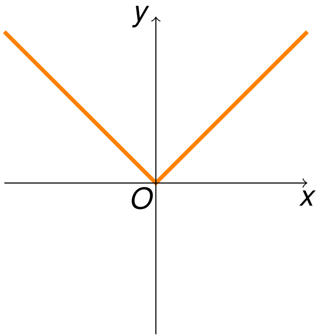


# Graph of the absolute value function



The absolute value function $f(x) = |x|$ is essentially the piecewise defined function

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



# Continuous functions



Intuitively, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called continuous if its graph has no "breaks" or "holes".

It should be noted that the above description is not the definition of continuity. As the most basic concept in calculus, the continuity of a real-valued function is strictly defined as:

## Definition

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x) - f(x_0)| < \epsilon.$$

$f$ is said to be continuous if it is continuous at any $x \in \mathbb{R}$.

# The vertical line test



The graph of a function is a curve in the coordinate plane. But the question arises: Which curves in the coordinate plane are graphs of functions? This is answered by the vertical line test:

**Proposition**

*A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.*

# Equations that define functions



**Example**

Does the equation define $y$ as a function of $x$?

(1) $y - x^2 = 2$.

(2) $x^2 + y^2 = 4$.

# Equations that define functions



**Solution.**

(1) $y = x^2 + 2$. It gives one value $y$ for each value of $x$, so it defines $y$ as a function of $x$.

(2) $y = \pm\sqrt{4 - x^2}$. It gives two values of $y$ for each $x \in (-2, 2)$, so it does not define $y$ as a function of $x$.

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