Fundamentals of Mathematics (MATH 1510)

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Outline

Vertical shifts of graphs

Suppose $c > 0$.

- To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.
- To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.

Vertical shifts of graphs

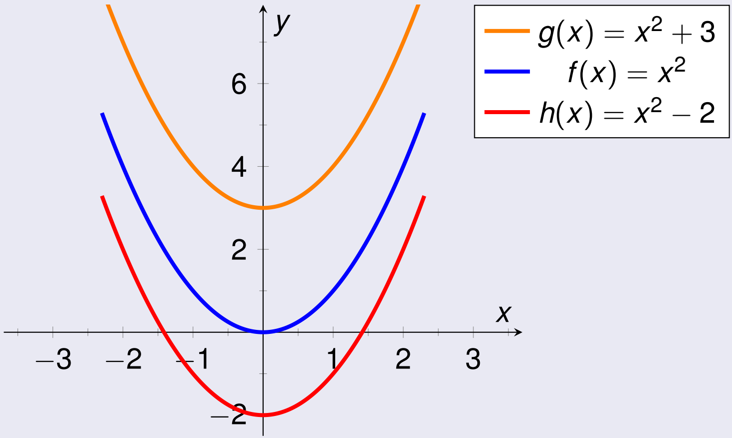
Example

Use the graph of $f(x) = x^2$ to sketch the graph of

$$g(x) = x^2 + 3 \quad \text{and} \quad h(x) = x^2 - 2.$$

Vertical shifts of graphs

Solution.



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Horizontal shifts of graphs

Suppose $c > 0$.

- To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.
- To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.

Horizontal shifts of graphs

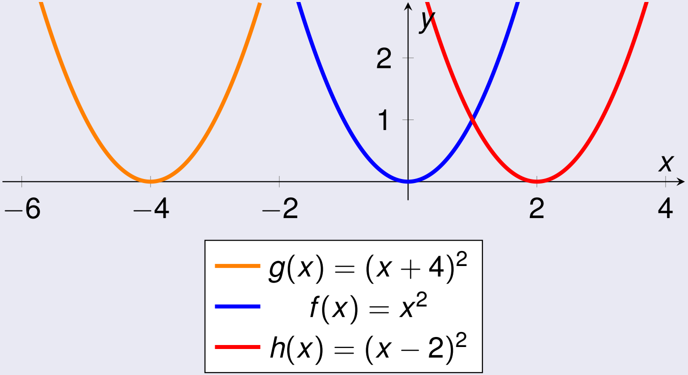
Example

Use the graph of $f(x) = x^2$ to sketch the graph of

$$g(x) = (x+4)^2 \quad \text{and} \quad h(x) = (x-2)^2.$$

Horizontal shifts of graphs

Solution.



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Reflecting graphs

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.

Reflecting graphs

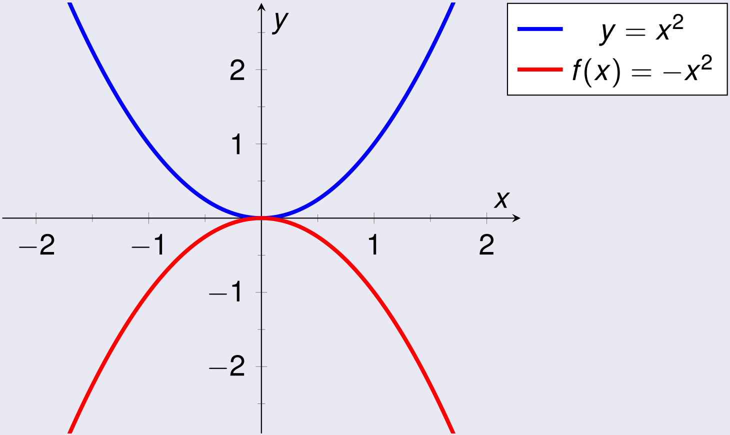
Example

Sketch the graph of

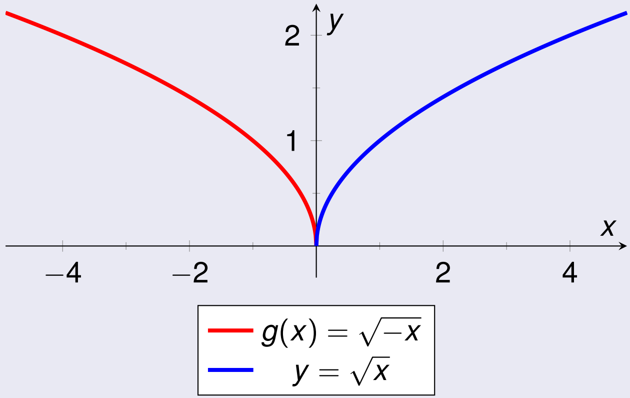
$$f(x) = -x^2 \quad \text{and} \quad g(x) = \sqrt{-x}.$$

Reflecting graphs

Solution.



Reflecting graphs



Vertical stretching and shrinking

Suppose $c > 0$. To graph $y = cf(x)$:

- If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.
- If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.

Vertical stretching and shrinking

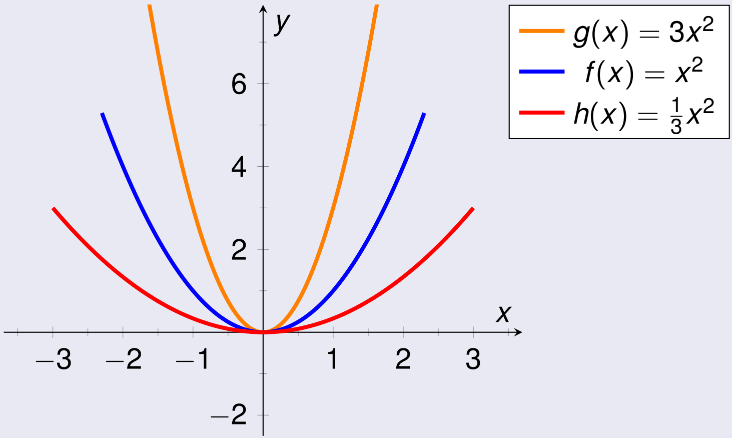
Example

Use the graph of $f(x) = x^2$ to sketch the graph of

$$g(x) = 3x^2 \quad \text{and} \quad h(x) = \frac{1}{3}x^2.$$

Vertical stretching and shrinking

Solution.



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Horizontal stretching and shrinking

Suppose $c > 0$. To graph $y = f(cx)$:

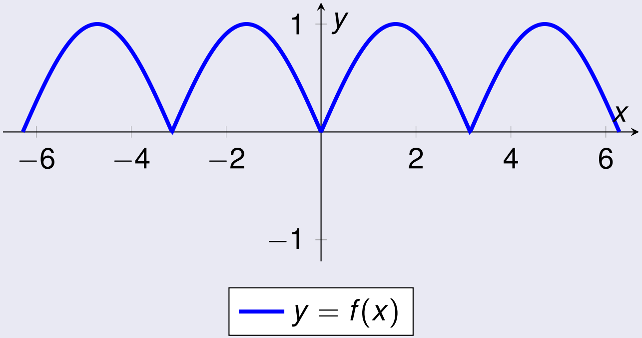
- If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{c}$.
- If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{c}$.

Horizontal stretching and shrinking

Example

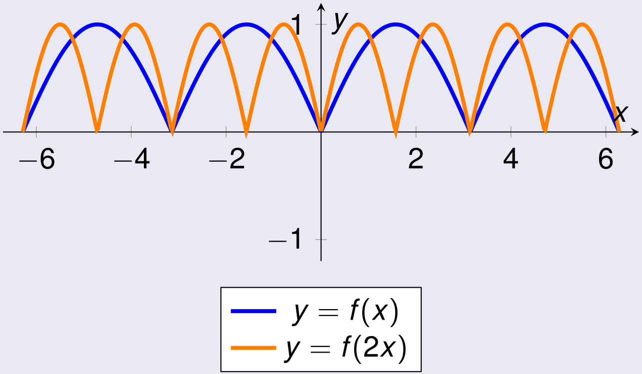
The graph of $y = f(x)$ is shown below. Sketch the graph of

$$y = f(2x) \quad \text{and} \quad y = f\left(\frac{1}{2}x\right).$$

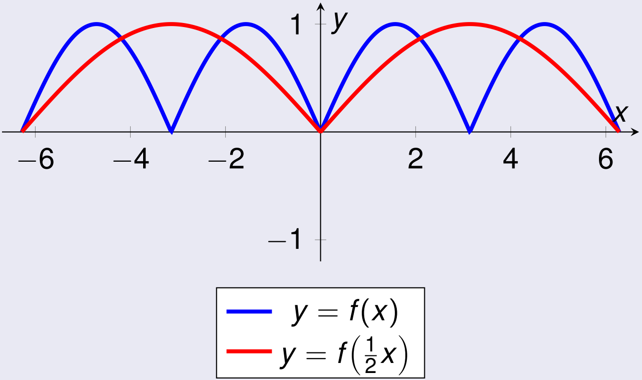


Horizontal stretching and shrinking

Solution.



Horizontal stretching and shrinking



Even and odd functions

Definition

Let f be a real-valued function.

- f is even if $f(-x) = f(x)$ for all x in the domain of f.
- f is odd if $f(-x) = -f(x)$ for all x in the domain of f.

The graph of an even function is symmetric with respect to the y-axis, and the graph of an odd function is symmetric with respect to the origin.

Even and odd functions

Example

Determine whether the following functions are even or odd.

(1) $f(x) = x^5 + x$.

(2) $g(x) = 1 - x^4$.

(3) $h(x) = 2x - x^2$.

Even and odd functions

Proof.

Determine whether the following functions are even or odd.

(1) $f(-x) = -x^5 - x = -f(x)$. Therefore f is odd.

(2) $g(-x) = 1 - x^4 = g(x)$. Therefore g is even.

(3) $h(-x) = -2x - x^2$. Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, h is neither even nor odd.

□

Outline

Sums, differences, products and quotients

Sums, differences, products and quotients of functions are defined pointwise; that is,

$$\begin{aligned}(f \pm g)(x) &= f(x) \pm g(x),\\ (fg)(x) &= f(x)g(x),\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}.\end{aligned}$$

Sums, differences, products and quotients

Example

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$. Find $f+g$, $f-g$, fg, $\dfrac{f}{g}$ and their domains.

Sums, differences, products and quotients

Solution.

$$(f+g)(x) = \frac{1}{x-2} + \sqrt{x}, \qquad x \in [0,2) \cup (2,\infty),$$

$$(f-g)(x) = \frac{1}{x-2} - \sqrt{x}, \qquad x \in [0,2) \cup (2,\infty),$$

$$(fg)(x) = \frac{\sqrt{x}}{x-2}, \qquad x \in [0,2) \cup (2,\infty),$$

$$\left(\frac{f}{g}\right)(x) = \frac{1}{(x-2)\sqrt{x}}, \qquad x \in (0,2) \cup (2,\infty).$$

□

Composition of functions

Definition

Given two functions

$$f : A \longrightarrow B \quad \text{and} \quad g : B \longrightarrow C,$$

the composition

$$g \circ f : A \longrightarrow C$$

is defined by

$$(g \circ f)(x) = g(f(x))$$

for all $x \in A$.

Composition of functions

Example

Let $f(x) = x^2$ and $g(x) = x - 3$. Find $f \circ g$ and $g \circ f$ and their domains.

Composition of functions

Solution.

$$(f \circ g)(x) = f(x-3) = (x-3)^2,$$
$$(g \circ f)(x) = g(x^2) = x^2 - 3.$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$. □

Composition of functions

Example

Find $f \circ g \circ h$ if $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$, $h(x) = x + 3$.

Composition of functions

Proof.

$$\begin{aligned}(f \circ g \circ h) &= (f \circ g)(x+3) \\ &= f((x+3)^{10}) \\ &= \frac{(x+3)^{10}}{(x+3)^{10}+1}.\end{aligned}$$

□

Outline

One-to-one functions

Definition

A function $f : A \longrightarrow B$ is called injective, or one-to-one, if no two elements of A have the same image; that is,

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2$$

for all $x_1, x_2 \in A$.

Horizontal line test

Proposition

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

One-to-one functions

Example

Is the function $f(x) = x^3$ one-to-one?

One-to-one functions

Solution.

Suppose $x_1^3 = x_2^3$. Since two different real numbers cannot have the same cube, one has $x_1 = x_2$, and it follows that $f(x) = x^3$ is one-to-one. $\square$

The inverse of a function

Definition

Let f be a one-to-one function with domain A and range B. Then its inverse function

$$f^{-1} : B \longrightarrow A$$

has domain B and range A with

$$f^{-1}(y) = x \iff f(x) = y$$

for all $y \in B$.

The inverse of a function

Remark

$f^{-1}(x)$ does not mean $\dfrac{1}{f(x)}$. The reciprocal $\dfrac{1}{f(x)}$ is written as

$$(f(x))^{-1}.$$

Inverse function property

The following proposition indicates that f and f^{-1} are inverses of each other:

Proposition

Let f be a one-to-one function with domain A and range B. Then

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

for all $x \in A$ and $y \in B$.

Conversely, any function $g : B \longrightarrow A$ satisfying

$$g(f(x)) = x \quad \text{and} \quad f(g(y)) = y$$

for all $x \in A$ and $y \in B$ is the inverse of f.

Finding the inverse of a function

Example

Find the inverse of the following functions:

(1) $f(x) = 3x - 2$.

(2) $f(x) = \dfrac{2x+3}{x-1}$.

Finding the inverse of a function

Solution.

(1) Let $y = f(x) = 3x - 2$. Then

$$x = \frac{y+2}{3}.$$

Therefore, $f^{-1}(x) = \dfrac{x+2}{3}$ is the inverse of f.

Finding the inverse of a function

(2) Let $y = f(x) = \dfrac{2x+3}{x-1}$. Then

$$\begin{aligned} yx - y &= 2x + 3, \\ (y-2)x &= y + 3, \\ x &= \frac{y+3}{y-2}. \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{x+3}{x-2}$ is the inverse of f. □

Graphing the inverse of a function

Proposition

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Graphing the inverse of a function

Example

(1) Sketch the graph of $f(x) = \sqrt{x-2}$.

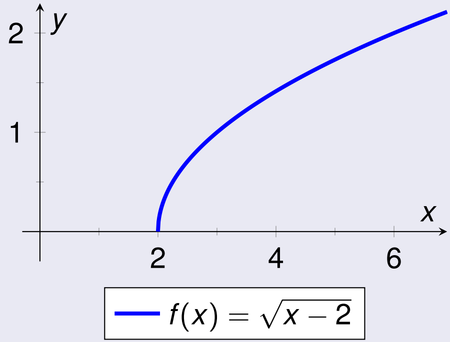
(2) Use the graph of f to sketch the graph of f^{-1}.

(3) Find an equation of f^{-1}.

Graphing the inverse of a function

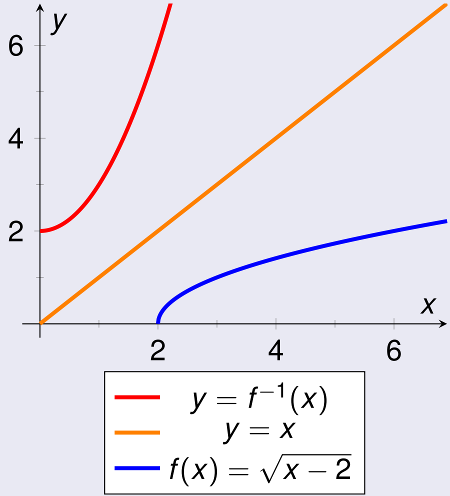
Solution.

(1)



Graphing the inverse of a function

(2)



Graphing the inverse of a function

(3) Solve $y = \sqrt{x-2}$ for x, noting that $y \geq 0$.

$$\begin{aligned} \sqrt{x-2} &= y, \\ x - 2 &= y^2, \\ x &= y^2 + 2. \end{aligned}$$

Therefore,

$$f^{-1}(x) = x^2 + 2 \quad (x \geq 0)$$

is the inverse of f.

□