Fundamentals of Mathematics (MATH 1510)

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Outline

Polynomial functions

Definition

A polynomial function of degree n is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_n \neq 0$.

- The numbers $a_0, a_1, a_2, \ldots, a_n$ are coefficients of the polynomial.
- The number a_0 is the constant coefficient or constant term.
- The number a_n is the leading coefficient, and $a_n x^n$ is the leading term.

Polynomial functions

Polynomial functions are usually called polynomials for short. The following polynomial has degree 5, leading coefficient 3, and constant term -6:

$$3x^5 + 6x^4 - 2x^3 + x^2 + 7x - 6.$$

A polynomial with only one term is a monomial, such as

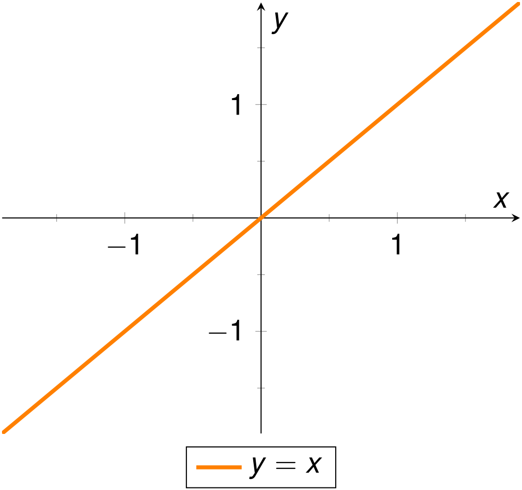
$$P(x) = x^3 \quad \text{and} \quad Q(x) = -6x^5.$$

Monomials

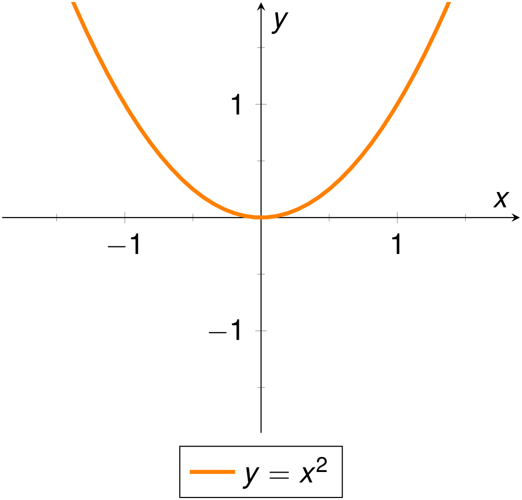
The simplest polynomial functions are monomials $P(x) = x^n$. It is easy to see that

- $P(x) = x^n$ is an odd function if n is an odd integer.
- $P(x) = x^n$ is an even function if n is an even integer.

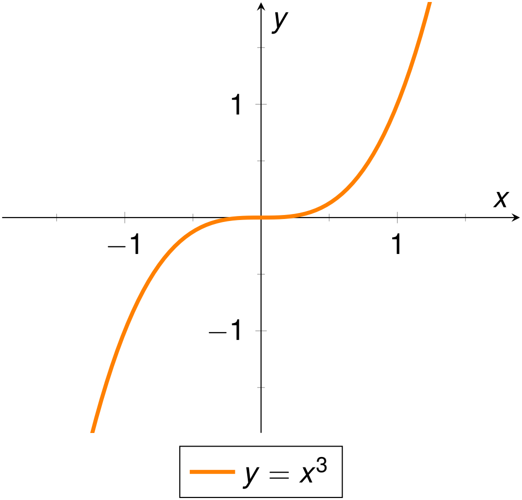
Graphs of monomials



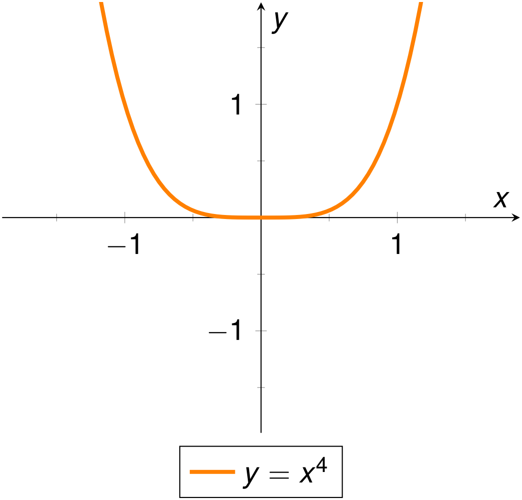
Graphs of monomials



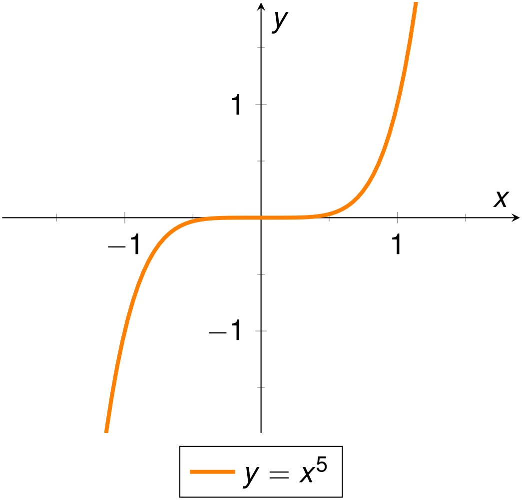
Graphs of monomials



Graphs of monomials



Graphs of monomials



Transformations of monomials

Example

Sketch the graphs of the following functions:

(1) $P(x) = -x^3$.

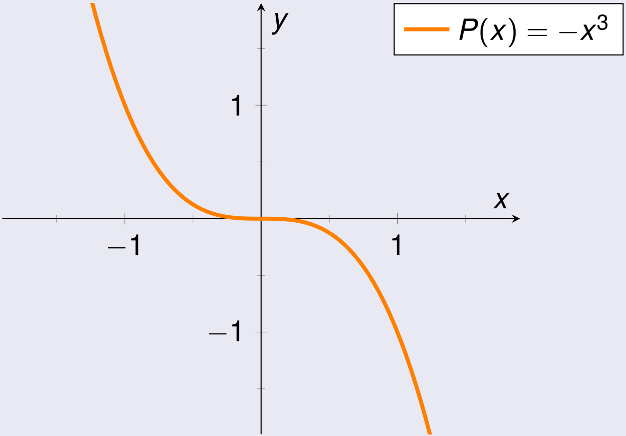
(2) $Q(x) = (x-2)^4$.

(3) $R(x) = -2x^5 + 4$.

Transformations of monomials

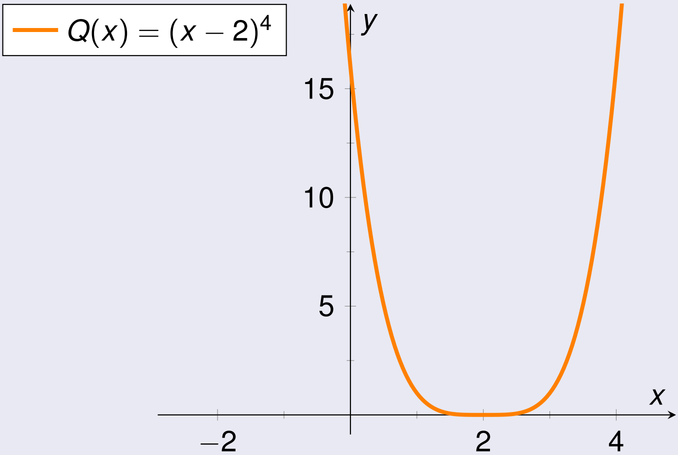
Solution.

(1)



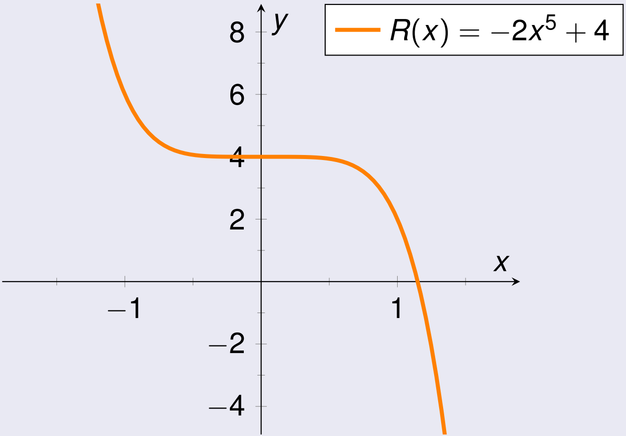
Transformations of monomials

(2)



Transformations of monomials

(3)



Graphs of polynomials

The domain of a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is always $\mathbb{R}$ and the graph of $P(x)$ is always smooth and continuous.

The rigorous definition of a function being “smooth” and “continuous” will be the topic of calculus.

End behaviour of polynomials

Proposition

The end behaviour of the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is determined by the degree n and the sign of the leading coefficient a_n:

- *If n is odd and a_n is positive, then $P(x) \to \infty$ as $x \to \infty$, and $P(x) \to -\infty$ as $x \to -\infty$.*
- *If n is odd and a_n is negative, then $P(x) \to -\infty$ as $x \to \infty$, and $P(x) \to \infty$ as $x \to -\infty$.*
- *If n is even and a_n is positive, then $P(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$.*
- *If n is even and a_n is negative, then $P(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$.*

End behaviour of polynomials

Example

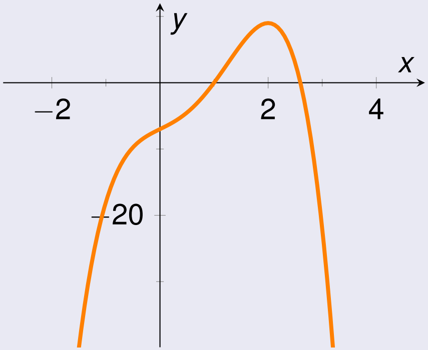
Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7.$$

End behaviour of polynomials

Solution.

The polynomial P has degree 4 and leading coefficient -2. So $P(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$.



□

Real zeros of polynomials

Proposition

If P is a polynomial and $c \in \mathbb{R}$, then the following statements are equivalent:

- *c is a zero of P.*
- *$x = c$ is a solution of the equation $P(x) = 0$.*
- *$x - c$ is a factor of $P(x)$.*
- *c is an x-intercept of the graph of P.*

Real zeros of polynomials

Example

If $P(x) = x^2 + x - 6$, then

- 2 is a zero of P.
- $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.
- $x - 2$ is a factor of $x^2 + x - 6$.
- 2 is an x-intercept of the graph of P.

The same facts are true for another zero, -3, of P.

Intermediate value theorem for polynomials

Theorem

Let P be a polynomial. If $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b such that

$$P(c) = 0.$$

Graphs of polynomials

Example

Let $P(x) = x^3 - 2x^2 - 3x$.

(1) Find the zeros of P.

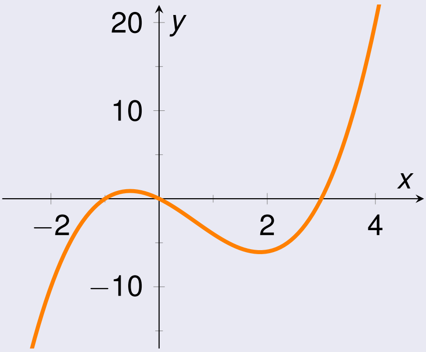
(2) Sketch the graph of P.

Graphs of polynomials

Solution.

(1) $P(x) = x(x-3)(x+1)$, so the zeros are $0, 3, -1$.

(2)



Shape of the graph near a zero

If c is a zero of a polynomial P and the corresponding factor $x - c$ occurs exactly m times in the factorization of P, then we say that c is a zero of multiplicity m.

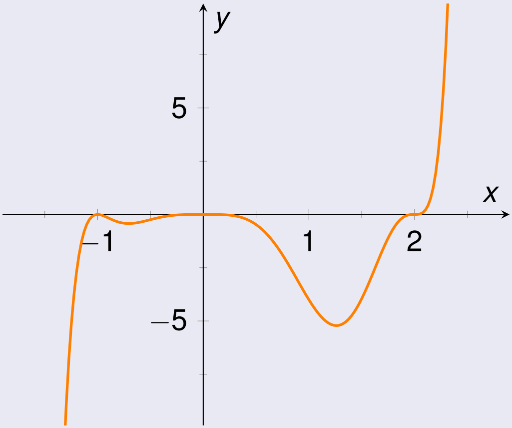
It can be shown by using calculus that near $x = c$ the graph has the general shape as the graph of

$$y = A(x - c)^m.$$

Shape of the graph near a zero

Example

The graph of $P(x) = x^4(x-2)^3(x+1)^2$ is sketched as:



Local extrema of functions

Definition

Let f be a function.

- If the point $(a, f(a))$ is the highest point on the graph of f within some range, then $(a, f(a))$ is a local maximum of f.
- If the point $(b, f(b))$ is the lowest point on the graph of f within some range, then $(b, f(b))$ is a local minimum of f.

Local maxima and local minima are both local extrema of f.

Local extrema of polynomials

Proposition

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then P has at most $n - 1$ local extrema.

A polynomial of degree n may in fact have fewer than $n - 1$ local extrema. For example, $P(x) = x^5$ has no local extrema, even though it is of degree 5.

Local extrema of polynomials

Example

$P(x) = x^4 + x^3 - 16x^2 - 4x + 48$ has three local extrema as its graph shows: