Fundamentals of Mathematics (MATH 1510)

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Outline

1. Dividing Polynomials
2. Rational Functions

Division algorithm

Definition

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$P(x) = D(x)Q(x) + R(x).$$

$P(x)$ and $Q(x)$ are called the dividend and divisor, respectively. $Q(x)$ is the quotient, and $R(x)$ is the remainder.

Long division of polynomials

Example

Divide $6x^2 - 26x + 12$ by $x - 4$.

Long division of polynomials

Solution.

$$
\begin{array}{rrrr}
 & & 6x & -\,2 \\
\cline{2-4}
x-4\,\big) & 6x^2 & -\,26x & +\,12 \\
 & -\,6x^2 & +\,24x & \\
\cline{2-3}
 & & -\,2x & +\,12 \\
 & & 2x & -\,8 \\
\cline{3-4}
 & & & 4
\end{array}
$$

Thus

$$6x^2 - 26x + 12 = (x-4)(6x-2) + 4.$$

□

Synthetic division

Example

Divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Synthetic division

Solution.

$$\begin{array}{r|rrrr} & 2 & -7 & 0 & 5 \\ 3 & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

Thus

$$2x^3 - 7x^2 + 5 = (x-3)(2x^2 - x - 3) - 4.$$

□

The remainder theorem

Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

The remainder theorem

Example

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(1) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(2) Use the remainder theorem to find $P(-2)$.

The remainder theorem

Solution.

(1)

	3	5	−4	0	7	3
−2		−6	2	4	−8	2
	3	−1	−2	4	−1	5

Thus the quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$, and the remainder is 5.

(2) $P(-2) = 5$.

□

The factor theorem

Theorem

$P(c) = 0$ if and only if $x - c$ is a factor of $P(x)$.

The factor theorem

Example

Factorize $P(x) = x^3 - 7x + 6$.

The factor theorem

Solution.

Since $P(1) = 0$, we know that $x - 1$ is a factor of $P(x)$.

$$\begin{array}{c|cccc} & 1 & 0 & -7 & 6 \\ 1 & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Thus

$$P(x) = (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3).$$

□

Outline

Rational functions

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. Usually we assume that $P(x)$ and $Q(x)$ have no common divisors.

The domain of a rational function consists of all real numbers x except those for which the denominator $Q(x) = 0$.

A simple rational function

Example

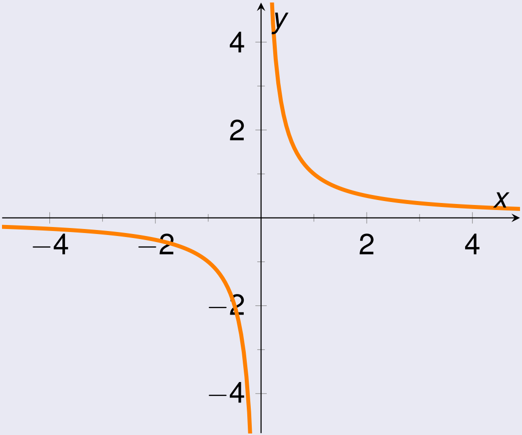
Consider the rational function $f(x) = \dfrac{1}{x}$. The domain and the range of f are both $\{x \mid x \neq 0\}$.

The behavior of f can be described as:

- $f(x) \to -\infty$ as $x \to 0^-$;
- $f(x) \to \infty$ as $x \to 0^+$;
- $f(x) \to 0$ as $x \to -\infty$ or $x \to \infty$.

A simple rational function

The graph of f is sketched as:



Vertical and horizontal asymptotes

The lines $x = 0$ and $y = 0$ are respectively the vertical asymptote and the horizontal asymptote of $f(x) = \dfrac{1}{x}$.

Informally speaking, an asymptote of a function is a line to which the graph of the function gets closer and closer as one travels along that line.

Vertical and horizontal asymptotes

Definition

- The line $x = a$ is a vertical asymptote of the function $y = f(x)$ if $y \to \infty$ or $y \to -\infty$ as $x \to a^+$ or $x \to a^-$.
- The line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if $y \to b$ as $x \to \infty$ or $x \to -\infty$.

Transformations of $y = 1/x$

A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be obtained by by shifting, stretching, and/or reflecting the function

$$f(x) = \frac{1}{x}.$$

Transformations of $y = 1/x$

Example

State their domain, range and asymptotes of the following functions, and sketch their graphs:

(1) $r(x) = \dfrac{2}{x-3}$.

(2) $s(x) = \dfrac{3x+5}{x+2}$.

Transformations of $y = 1/x$

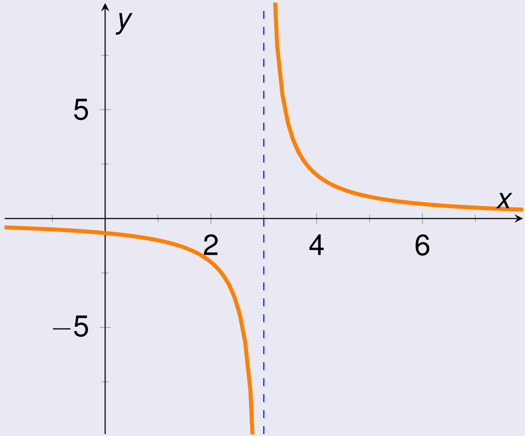
Solution.

(1) Let $f(x) = \dfrac{1}{x}$. Then

$$r(x) = \frac{2}{x-3} = 2\left(\frac{1}{x-3}\right) = 2f(x-3).$$

The domain is $\{x \mid x \neq 3\}$, and the range is $\{y \mid y \neq 0\}$. r has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$.

Transformations of $y = 1/x$



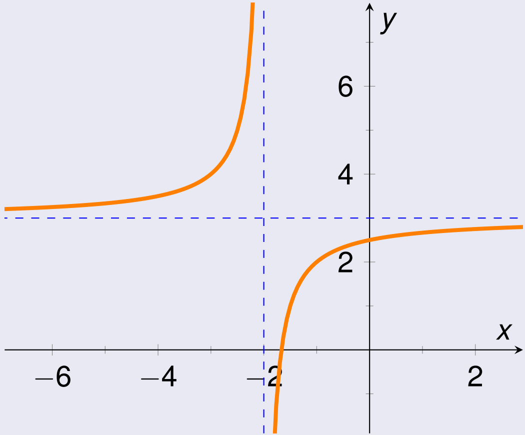
Transformations of $y = 1/x$

(2) Let $f(x) = \dfrac{1}{x}$. Then

$$s(x) = \frac{3x+5}{x+2} = 3 - \frac{1}{x+2} = -f(x+2) + 3.$$

The domain is $\{x \mid x \neq -2\}$, and the range is $\{y \mid y \neq 3\}$. r has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$.

Transformations of $y = 1/x$



Asymptotes of rational functions

Definition

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

whose numerator and denominator have no common factors.

- The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.
- If $n < m$, then r has horizontal asymptote $y = 0$.
- If $n = m$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$.
- If $n > m$, then r has no horizontal asymptote.

Asymptotes of rational functions

Example

Find the vertical and horizontal asymptotes of

$$r(x) = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}.$$

Asymptotes of rational functions

Solution.

Since

$$r(x) = \frac{3x^2 - 2x - 1}{(2x-1)(x+2)},$$

the vertical asymptotes are $x = \dfrac{1}{2}$ and $x = -2$, and the horizontal asymptote is $y = \dfrac{3}{2}$. □

Graphing a rational function

Example

Graph $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$.

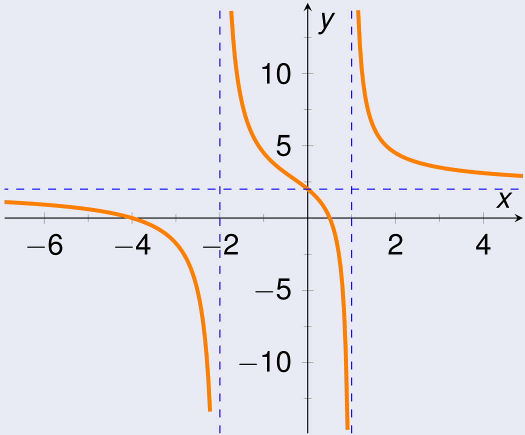
Graphing a rational function

Solution.

$$r(x) = \frac{(2x-1)(x+4)}{(x+2)(x-1)}.$$

- The x-intercepts are $x = \frac{1}{2}$ and $x = 4$.
- The y-intercept is $y = 2$.
- The vertical asymptotes are $x = 1$ and $x = -2$.
- $y \to -\infty$ as $x \to -2^-$, $y \to \infty$ as $x \to -2^+$.
- $y \to \infty$ as $x \to 1^-$, $y \to \infty$ as $x \to 1^+$.
- The horizontal asymptote is $y = 2$, i.e., $y \to 2$ as $x \to -\infty$ or $x \to \infty$.
- The domain of r is $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$ and the range is $\mathbb{R}$.

Graphing a rational function



Common factors in numerator and denominator

Example

Graph the following functions:

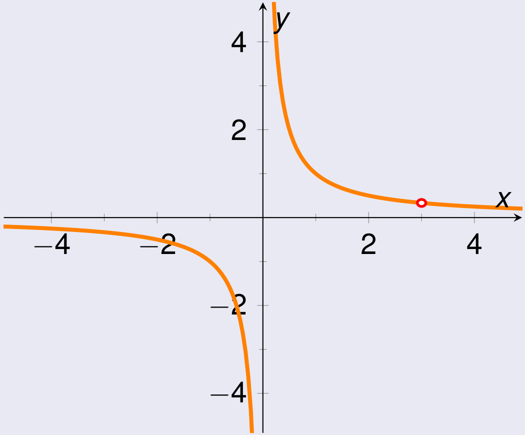
(1) $s(x) = \dfrac{x-3}{x^2-3x}$.

(2) $t(x) = \dfrac{x^3-2x^2}{x-2}$.

Common factors in numerator and denominator

Solution.

(1) $s(x) = \dfrac{x-3}{x^2-3x} = \dfrac{1}{x}$ $(x \neq 3)$.



Common factors in numerator and denominator

(2) $t(x) = \dfrac{x^3 - 2x^2}{x - 2} = x^2 \ (x \neq 2).$