Fundamentals of Mathematics (MATH 1510)

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Outline

Systems of linear equations

A linear equation in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c,$$

where $a_1, a_2, \ldots, a_n, c \in \mathbb{R}$ are constants, and $x_1, x_2, \ldots, x_n$ are variables.

A system of linear equations in n variables consists of several linear equations of the above form that involve the same variables.

Solving a linear system

A system in the form of

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ \qquad\quad\; a_{22}y + a_{23}z = c_2 \\ \qquad\qquad\qquad\; a_{33}z = c_3 \end{cases}$$

is in triangular form. It is easy to solve such a system by using back-substitution.

Solving a linear system

Example

Solve the system

$$\begin{cases} x - 2y - z = 1 & (1) \\ y + 2z = 5 & (2) \\ z = 3 & (3) \end{cases}$$

Solving a linear system

Solution.

From (3) we know $z = 3$. We back-substitute this into (2) and get

$$y = 5 - 2z = 5 - 2 \cdot 3 = -1.$$

Then we back-substitute $y = -1$ and $z = 3$ into (1) and get

$$x = 1 + z + 2y = 1 + 3 + 2(-1) = 2.$$

Therefore the solution of the system is

$$\begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

□

Gaussian elimination

The following elementary operations on a system of linear equations yield an equivalent system:

- Interchange the positions of two equations.
- Multiply an equation by a nonzero constant.
- Add a nonzero multiple of one equation to another.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. This process is called the Gaussian elimination.

Gaussian elimination

Example

Solve the system

$$\begin{cases} x - 2y + 3z = 1 & (1) \\ x + 2y - z = 13 & (2) \\ 3x + 2y - 5z = 3 & (3) \end{cases}$$

Gaussian elimination

Solution.

$$\begin{cases} x-2y+3z=1 \\ x+2y-\ \ z=13 \\ 3x+2y-5z=3 \end{cases} \Longrightarrow \begin{cases} x-2y+\ \ 3z=1 \\ \quad\ \ 4y-\ \ 4z=12 \\ \quad\ \ 8y-14z=0 \end{cases}$$

$$\Longrightarrow \begin{cases} x-2y+3z=1 \\ \quad\ \ 4y-4z=12 \\ \qquad\ -6z=-24 \end{cases} \Longrightarrow \begin{cases} x-2y+3z=1 \\ \quad\ \ y-\ \ z=3 \\ \qquad\quad z=4 \end{cases}$$

Thus $z=4$, $y=z+3=7$, $x=2y-3z+1=3$, and the solution of this system is

$$x=3,\ y=7,\ z=4.$$

□

The number of solutions

For a system of linear equations, exactly one of the following is true:

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system with no solution is said to be inconsistent, and a system with infinitely many solutions is said to be dependent.

A system with no solution

Example

Solve the system

$$\begin{cases} x + 2y - 2z = 1 & (1) \\ 2x + 2y - z = 6 & (2) \\ 3x + 4y - 3z = 5 & (3) \end{cases}$$

A system with no solution

Solution.

$$\begin{cases} x + 2y - 2z = 1 \\ 2x + 2y - z = 6 \\ 3x + 4y - 3z = 5 \end{cases} \Longrightarrow \begin{cases} x + 2y - 2z = 1 \\ -2y + 3z = 4 \\ -2y + 3z = 2 \end{cases}$$

$$\Longrightarrow \begin{cases} x + 2y - 2z = 1 \\ -2y + 3z = 4 \\ 0 = -2 \end{cases}$$

Thus this system has no solution. □

A system with infinitely many solutions

Example

Solve the system

$$\begin{cases} x - y + 5z = -2 & (1) \\ 2x + y + 4z = 2 & (2) \\ 2x + 4y - 2z = 8 & (3) \end{cases}$$

A system with infinitely many solutions

Solution.

$$\begin{cases} x - \ y + 5z = -2 \\ 2x + \ y + 4z = 2 \\ 2x + 4y - 2z = 8 \end{cases} \Longrightarrow \begin{cases} x - \ y + \ 5z = -2 \\ \quad 3y - \ 6z = 6 \\ \quad 6y - 12z = 12 \end{cases}$$

$$\Longrightarrow \begin{cases} x - \ y + 5z = -2 \\ \quad 3y - 6z = 6 \\ \quad\quad 0 = 0 \end{cases} \Longrightarrow \begin{cases} x - y + 5z = -2 \\ \quad y - 2z = 2 \end{cases}$$

The solutions of this system are

$$x = -3t, \ y = 2t + 2, \ z = t,$$

where t can be any real number. $\square$

Outline

Substitution method

Example

Solve the system

$$\begin{cases} x^2 + y^2 = 100 & (1) \\ 3x - \ \ y = 10 & (2) \end{cases}$$

Substitution method

Solution.

From (2) one has $y = 3x - 10$, and it follows from (1) that

$$\begin{aligned} x^2 + (3x-10)^2 &= 100, \\ 10x^2 - 60x &= 0, \\ x(x-6) &= 0. \end{aligned}$$

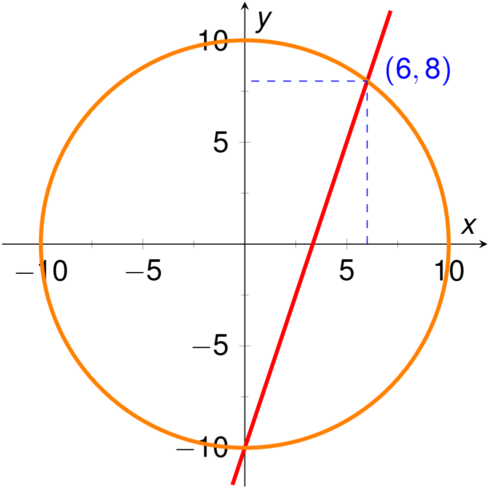
Thus $x = 0$ or $x = 6$, and consequently

$$\begin{cases} x = 0 \\ y = -10 \end{cases} \quad \text{and} \quad \begin{cases} x = 6 \\ y = 8 \end{cases}$$

are the solutions of the system. □

Substitution method

The graphs of $x^2 + y^2 = 100$ and $3x - y = 10$ confirm our solutions:



Elimination method

Example

Solve the system

$$\begin{cases} 3x^2 + 2y = 26 & (1) \\ 5x^2 + 7y = 3 & (2) \end{cases}$$

Elimination method

Solution.

By $5 \cdot (1) - 3 \cdot (2)$ one has

$$-11y = 121$$

and thus $y = -11$. Hence

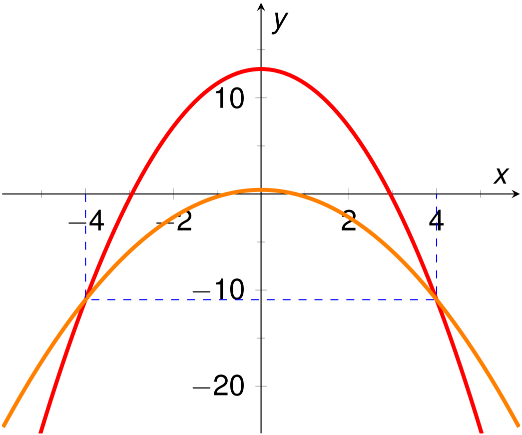
$$3x^2 = 26 - 2y = 48$$

and consequently $x = \pm 4$. Therefore,

$$\begin{cases} x = 4 \\ y = -11 \end{cases} \quad \text{and} \quad \begin{cases} x = -4 \\ y = -11 \end{cases}$$

are the solutions of the system. □

The graphs of $3x^2 + 2y = 26$ and $5x^2 + 7y = 3$ confirm our solutions:



Elimination method

Example

Solve the system

$$\begin{cases} x^2 - y = 2 & (1) \\ 2x - y = -1 & (2) \end{cases}$$

Elimination method

Solution.

By (1) − (2) one has

$$\begin{aligned} x^2 - 2x - 3 &= 0, \\ (x-3)(x+1) &= 0, \end{aligned}$$

and thus $x = 3$ or $x = -1$. Therefore,

$$\begin{cases} x = 3 \\ y = 3^2 - 2 = 7 \end{cases} \quad \text{and} \quad \begin{cases} x = -1 \\ y = 1 - 2 = -1 \end{cases}$$

are the solutions of the system. □

Elimination method

The graphs of $x^2 - y = 2$ and $2x - y = -1$ confirm our solutions: