Fundamentals of Mathematics (MATH 1510)

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Outline

1. Exponential Functions

Exponential functions

Functions of the form

$$f(x) = 2^x$$

are called exponential functions.

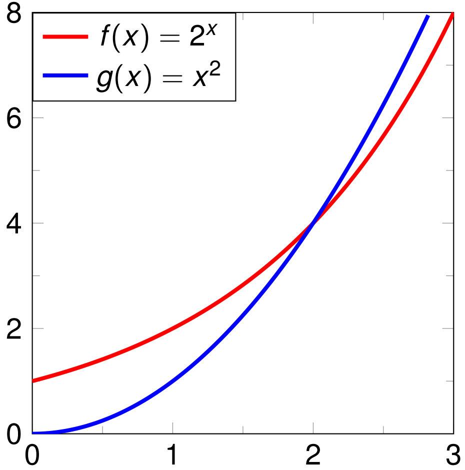
When the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function:

$$f(3) = 2^3 = 8,$$
$$f(10) = 2^{10} = 1024,$$
$$f(20) = 2^{20} = 1,048,576.$$

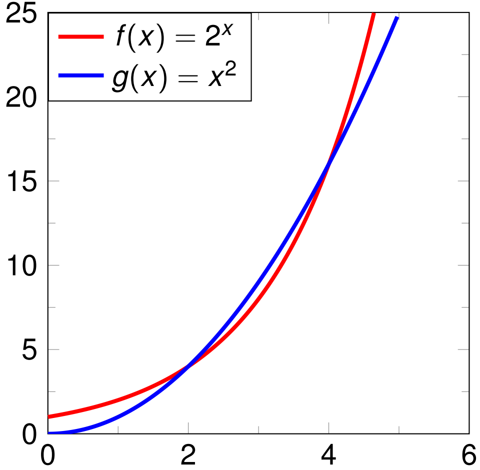
As a comparison, for the function $g(x) = x^2$ we have

$$g(3) = 3^2 = 9, \quad g(10) = 10^2 = 100, \quad g(20) = 20^2 = 400.$$

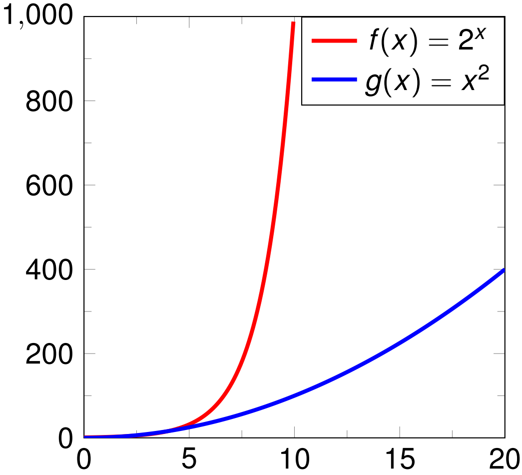
Comparing exponential and power functions



Comparing exponential and power functions



Comparing exponential and power functions



Exponential functions

Using the language of calculus, this property of exponential functions may be expressed as

$$\lim_{x\to\infty} \frac{x^n}{a^x} = 0$$

for all $a > 1$ and $n \in \mathbb{N}$, no matter how large n is or how small a is (as long as $a > 1$). For example,

$$\lim_{x\to\infty} \frac{x^{100}}{1.01^x} = 0.$$

Irrational exponents

To study exponential functions, we must define what we mean by the exponential expression a^x when x is any real number.

Recall that for any $a \geq 0$ and $q = \dfrac{m}{n} \in \mathbb{Q}$ with $n > 0$,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

To define a^x when x is irrational, we approximate x by rational numbers; that is, we find an infinite sequence of rational numbers $\{q_n\}$ with $q_n \to q$ $(n \to \infty)$, and define

$$a^x = \lim_{n\to\infty} a^{q_n}.$$

Irrational exponents

For example, since

$$\pi = 3.1415926535\ldots$$

is an irrational number, we successively approximate a^π by the following rational powers:

$$a^{3.1},\ a^{3.14},\ a^{3.141},\ a^{3.1415},\ a^{3.14159},\ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to a^π. It can be shown in calculus that there is exactly one number that these powers approach, which is defined as the value of a^π.

Laws of exponents

The following laws of exponents are still true when the exponents are real numbers:

Proposition

(1) $a^x a^y = a^{x+y}$.

(2) $\dfrac{a^x}{a^y} = a^{x-y}$.

(3) $(a^x)^y = a^{xy}$.

(4) $(ab)^x = a^x b^x$.

(5) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.

(6) $\left(\dfrac{a}{b}\right)^{-x} = \dfrac{b^x}{a^x}$.

(7) $\dfrac{a^{-y}}{b^{-x}} = \dfrac{b^x}{a^y}$.

Exponential functions

Definition

The exponential function with base a is defined for all real numbers x by

$$f(x) = a^x,$$

where $a > 0$ and $a \neq 1$.

It is easy to see that the expression a^x makes sense for all $x \in \mathbb{R}$ only when $a > 0$, and we assume $a \neq 1$ to eliminate the trivial case.

Calculating exponentials

Exponentials are usually denoted by the symbol

^ (Shift+6 in the standard keyboard)

in computers. For example, typing

3^pi

in Google and you will get

$$3^{\pi} \approx 31.5442807002.$$

Graphs of exponential functions

Graphs of exponential functions have an easily recognizable shape.

Example

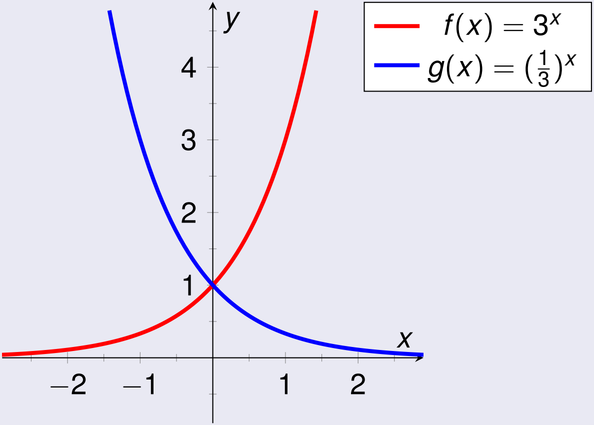
Draw the graph of each function.

(1) $f(x) = 3^x$.

(2) $g(x) = \left(\frac{1}{3}\right)^x$.

Graphs of exponential functions

Solution.



□

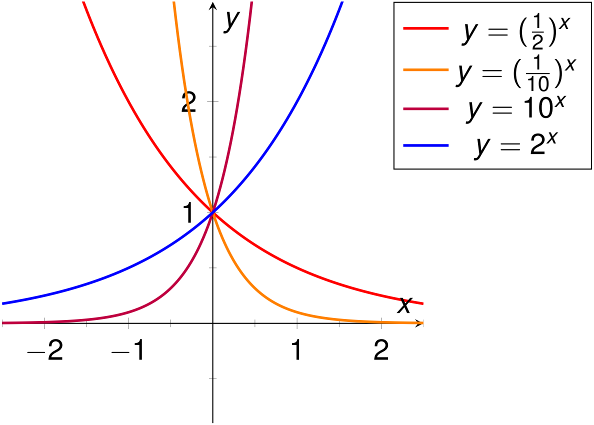
Graphs of exponential functions

It can be inferred from the graphs that the graph of g can be obtained from the graph of f by reflecting in the y-axis. This is true since

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x).$$

Graphs of exponential functions

More graphs of exponential functions are sketched below:



Graphs of exponential functions

The graph of every exponential function passes through the point $(0, 1)$ because $a^0 = 1$ for any $a \neq 0$.

- If $0 < a < 1$, the function $f(x) = a^x$ decreases rapidly.
- If $a > 1$, the function $f(x) = a^x$ increases rapidly.

$y = 0$ (the x-axis) is a horizontal asymptote for every exponential function $f(x) = a^x$. This is because

- if $a > 1$, then $a^x \to 0$ as $x \to -\infty$;
- if $0 < a < 1$, then $a^x \to 0$ as $x \to \infty$.

Since $a^x > 0$ for all $x \in \mathbb{R}$, the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

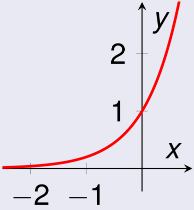
Graphs of exponential functions

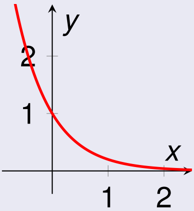
Proposition

For any $a > 0$ with $a \neq 1$, the exponential function

$$f(x) = a^x$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes:

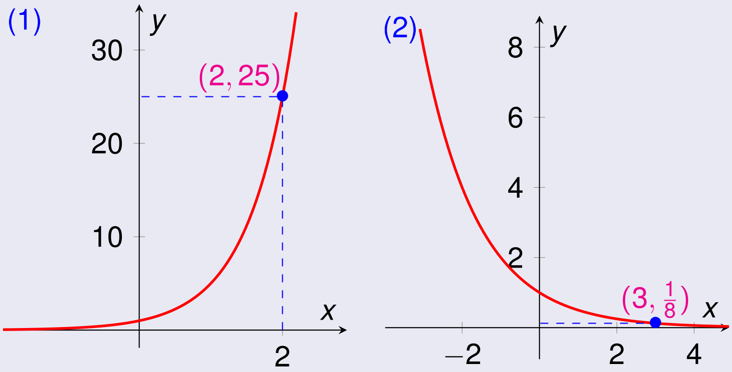




Identifying graphs of exponential functions

Example

Find the exponential function $f(x) = a^x$ whose graph is given.



Identifying graphs of exponential functions

Solution.

(1) Since $f(2) = a^2 = 25$, it follows that $a = 5$. So $f(x) = 5^x$.

(2) Since $f(3) = a^3 = \dfrac{1}{8}$, it follows that $a = \dfrac{1}{2}$. So $f(x) = \left(\dfrac{1}{2}\right)^x$.

□

Compound interest

Exponential functions have important applications in calculating compound interests. If an amount of money P, called the principal, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount of money becomes

$$A_1 = P + Pi = P(1 + i).$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period becomes

$$A_2 = A_1(1 + i) = P(1 + i)^2.$$

Compound interest

In general, after k periods the amount of money becomes

$$A = P(1+i)^k.$$

Note that this is an exponential function with base $1+i$.

If the annual interest rate is r and the interest is compounded n times per year, then in each time period the interest rate is

$$i = \frac{r}{n},$$

and there are nt time periods in t years.

Compound interest

Proposition

The compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where

$$\begin{aligned} A(t) &= \text{amount after } t \text{ years}, \\ P &= \text{principle}, \\ r &= \text{interest rate per year}, \\ n &= \text{number of times interest is compounded per year}, \\ t &= \text{number of years}. \end{aligned}$$

Calculating compound interest

Example

A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

Calculating compound interest

Solution.

Compounding	n	Amount after 3 years
Annual	1	$1000\left(1+\frac{0.12}{1}\right)^{3}=1404.93$
Semiannual	2	$1000\left(1+\frac{0.12}{2}\right)^{2\cdot 3}=1418.52$
Quarterly	4	$1000\left(1+\frac{0.12}{4}\right)^{4\cdot 3}=1425.76$
Monthly	12	$1000\left(1+\frac{0.12}{12}\right)^{12\cdot 3}=1430.77$
Daily	365	$1000\left(1+\frac{0.12}{365}\right)^{365\cdot 3}=1433.24$

□

Annual percentage yield

If an investment earns compound interest, then the annual percentage yield (APY) is the simple interest rate that yields the same amount at the end of one year.

Example

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

Annual percentage yield

Solution.

After one year, a principle P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = 1.06183P.$$

Since the formula for simple interest is

$$A = P(1 + r),$$

we see that the annual percentage yield is
$r = 0.06183 = 6.183\%$. □