Fundamentals of Mathematics (MATH 1510)

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Outline

1 The Natural Exponential Function

2 Logarithmic Functions

The number e

The (irrational) number e is defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

The approximate value of e is

$$e \approx 2.71828.$$

The natural exponential function

Definition

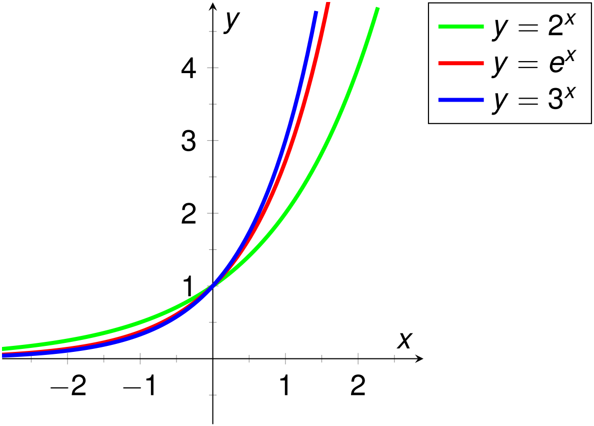
The natural exponential function is the exponential function

$$f(x) = e^x$$

with base e.

The natural exponential function

Since $2 < e < 3$, the graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$:



Continuously compounded interest

Recall that the compound interest is calculated by the formula

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= P\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} \\ &= P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}. \qquad \left(\text{letting } m=\frac{m}{r}\right) \end{aligned}$$

Therefore, as n becomes large, $A(t)$ approaches to Pe^{rt}, i.e.,

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt}.$$

Continuously compounded interest

Proposition

The continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt},$$

where

$$\begin{aligned} A(t) &= \text{amount after } t \text{ years}, \\ P &= \text{principle}, \\ r &= \text{interest rate per year}, \\ t &= \text{number of years}. \end{aligned}$$

Intuitively, the continuously compounded interest gives the interest compounded at “every instant”.

Continuously compounded interest

Example

Find the amount after 3 years if $1000 is invested at an interest rate of 12% per year, compounded continuously.

Continuously compounded interest

Solution.

$$A(3) = 1000e^{3\cdot 0.12} = 1000e^{0.36} \approx 1433.33.$$

□

Outline

1 The Natural Exponential Function

2 Logarithmic Functions

Logarithmic functions

Definition

Let $a \in \mathbb{R}$ with $a > 0$ and $a \neq 1$. The logarithmic function with base a, denoted by $\log_a$, is defined by

$$\log_a x = y \iff a^y = x.$$

So, $\log_a x$ is the exponent to which the base a must be raised to give x.

Logarithmic and exponential forms

When we use the definition of logarithms to switch back and forth between the logarithmic form $\log_a x = y$ and the exponential form $a^y = x$, it is helpful to notice that, in both forms, the base is the same.

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. For example:

Logarithmic form	Exponential form
$\log_{10} 100,000 = 5$	$10^5 = 100,000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2 \frac{1}{8} = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

Evaluating logarithms

It is important to understand that $\log_a x$ is an exponent:

- $\log_{10} 1000 = 3$ because $10^3 = 1000$.
- $\log_2 32 = 5$ because $2^5 = 32$.
- $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$.
- $\log_{16} 4 = \frac{1}{2}$ because $16^{\frac{1}{2}} = 4$.

Properties of logarithms

Proposition

- $\log_a 1 = 0$.
- $\log_a a = 1$.
- $\log_a a^x = x$ *for all* $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ *for all* $x > 0$.

Properties of logarithms

For example:

- $\log_5 1 = 0$.
- $\log_5 5 = 1$.
- $\log_5 5^8 = 8$.
- $5^{\log_5 12} = 12$.

Domains and ranges of logarithmic functions

Since the exponential functions $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function,

$$f^{-1}(x) = \log_a x,$$

has domain $(0, \infty)$ and range $\mathbb{R}$.

Domains and ranges of logarithmic functions

Example

Find the domain of the function $f(x) = \log_2(4 - x^2)$.

Domains and ranges of logarithmic functions

Solution.

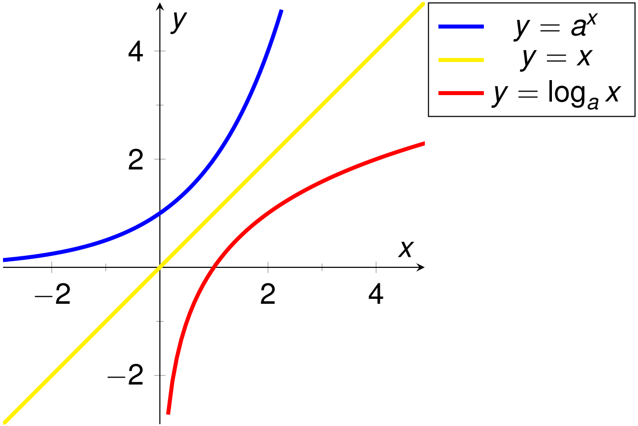
Since $\log_2 x$ is defined when $x > 0$, the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = (-2, 2).$$

□

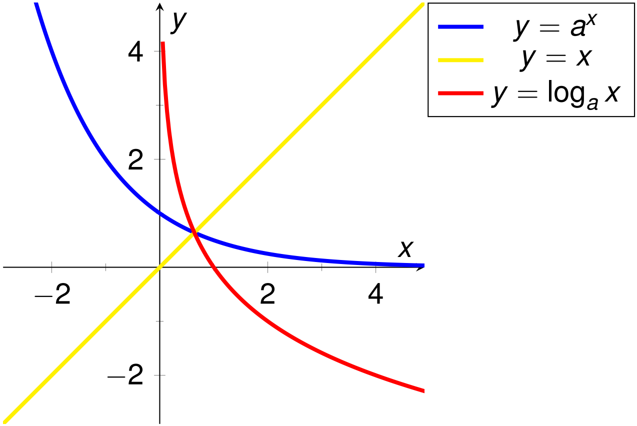
Graphs of logarithmic functions

The graph of $y = \log_a x$ is obtained by reflecting the graph of $y = a^x$ in the line $y = x$. If $a > 1$:



Graphs of logarithmic functions

If $0 < a < 1$:



Graphs of logarithmic functions

The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

Using the language of calculus, this property of logarithmic functions may be expressed as

$$\lim_{x \to \infty} \frac{\log_a x}{x^b} = 0$$

for all $a > 1$ and $b \in \mathbb{R}^+$, no matter how large a is or how small b is (as long as $b > 0$).

Graphs of logarithmic functions

Since $\log_a 1 = 0$, the x-intercept of $y = \log_a x$ is always 1.

The y-axis (i.e., $x = 0$) is a vertical asymptote of $y = \log_a x$ since

- if $a > 1$, $\log_a x \to -\infty$ as $x \to 0^+$,
- if $0 < a < 1$, $\log_a x \to \infty$ as $x \to 0^+$.

Graphs of logarithmic functions

Example

Sketch the graph of each function. State the domain, range, and asymptote:

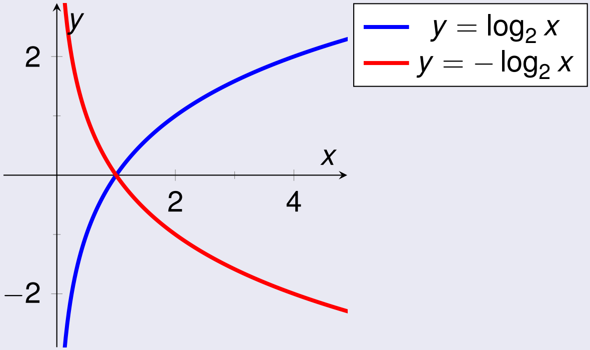
(1) $g(x) = -\log_2 x$.

(2) $h(x) = \log_2(-x)$.

Graphs of logarithmic functions

Solution.

(1)

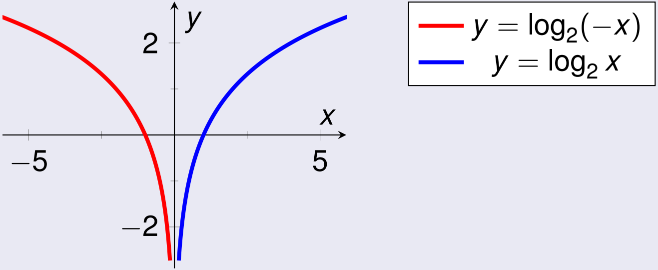


The domain is $(0, \infty)$, the range is $\mathbb{R}$ and $x = 0$ is the vertical asymptote.

Graphs of logarithmic functions

Solution.

(2)



The domain is $(-\infty, 0)$, the range is $\mathbb{R}$ and $x = 0$ is the vertical asymptote.

Common logarithms

Definition

The logarithm with base 10 is called the common logarithm and is denoted by omitting the base:

$$\log x = \log_{10} x.$$

Evaluating the common logarithm function

Example

- $\log 0.01 = -2.$
- $\log 0.1 = -1.$
- $\log 0.5 \approx -0.301.$
- $\log 1 = 0.$
- $\log 5 \approx 0.699.$
- $\log 10 = 1.$
- $\log 100 = \log 10^2 = 2.$
- $\log 50 \approx 1.699.$

Common logarithms

Example

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10 \log \left(\frac{I}{I_0} \right),$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

Common logarithms

Solution.

$$B = 10\log\left(\frac{I}{I_0}\right) = 10\log 100 = 20.$$

The loudness of the sound is 20 dB. □

Natural logarithms

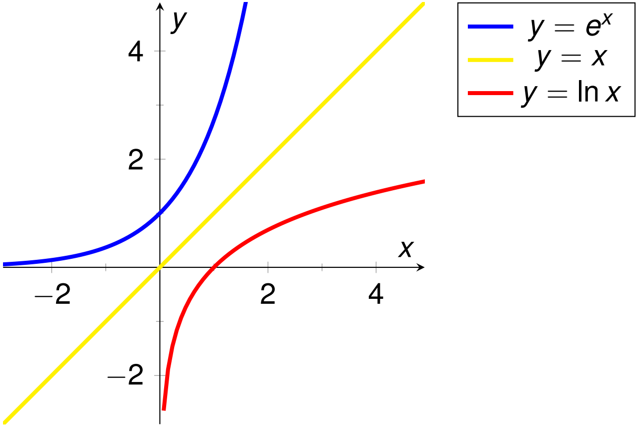
Definition

The logarithm with base e is called the natural logarithm and is denoted by

$$\ln x = \log_e x.$$

Natural logarithms

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$:



Properties of natural logarithms

Proposition

- $\ln 1 = 0$.
- $\ln e = 1$.
- $\ln e^x = x$ *for all* $x \in \mathbb{R}$.
- $e^{\ln x} = x$ *for all* $x > 0$.

Evaluating the natural logarithm function

Example

- $\ln e^8 = 8$.
- $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$.
- $\ln 5 \approx 1.609$.