Fundamentals of Mathematics (MATH 1510)

Instructor: Lili Shen
Email: shenlili@yorku.ca

Department of Mathematics and Statistics
York University

January 18-22, 2016

Outline

Laws of logarithms

Proposition

Let $a > 0$ *with* $a \neq 1$. *Let* $A, B, C \in \mathbb{R}$ *with* $A > 0$ *and* $B > 0$.

(1) $\log_a(AB) = \log_a A + \log_a B$.

(2) $\log_a \frac{A}{B} = \log_a A - \log_a B$.

(3) $\log_a(A^C) = C \log_a A$.

Laws of logarithms

Example

Evaluate each expression.

(1) $\log_4 2 + \log_4 32.$

(2) $\log_2 80 - \log_2 5.$

(3) $-\frac{1}{3} \log 8.$

Laws of logarithms

Solution.

(1) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3.$

(2) $\log_2 80 - \log_2 5 = \log_2 \dfrac{80}{5} = \log_2 16 = 4.$

(3) $-\dfrac{1}{3}\log 8 = \log 8^{-\frac{1}{3}} = \log \dfrac{1}{2} \approx -0.301.$

□

Expanding logarithmic expressions

Example

Expand each expression:

(1) $\log_2(6x)$.

(2) $\log_5(x^3y^6)$.

(3) $\ln \dfrac{ab}{\sqrt[3]{c}}$.

Expanding logarithmic expressions

Solution.

(1) $\log_2(6x) = \log_2 6 + \log_2 x$.

(2) $\log_5(x^3 y^6) = \log_6 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$.

(3) $\ln \dfrac{ab}{\sqrt[3]{c}} = \ln(ab) - \ln \sqrt[3]{c} = \ln a + \ln b - \dfrac{1}{3}\ln c$.

□

Combining logarithmic expressions

Example

Combine each expression into a single logarithm:

(1) $3\log x + \frac{1}{2}\log(x+1)$.

(2) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$.

Combining logarithmic expressions

Solution.

(1)

$$3\log x + \frac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{\frac{1}{2}}$$
$$= \log(x^3\sqrt{x+1}).$$

(2)

$$3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{\frac{1}{2}} - \ln(t^2+1)^4$$
$$= \ln\frac{s^3\sqrt{t}}{(t^2+1)^4}.$$

□

Combining logarithmic expressions

Example

Combine and simplify

$$\frac{1}{5}\log(x+2)^5+\frac{1}{3}[\log(x^6)-\log(x^2-x-6)^3].$$

Combining logarithmic expressions

Solution.

$$
\begin{aligned}
&\frac{1}{5}\log(x+2)^5 + \frac{1}{3}[\log(x^6) - \log(x^2-x-6)^3] \\
&= \log(x+2) + \log(x^2) - \log(x^2-x-6) \\
&= \log\frac{x^2(x+2)}{x^2-x-6} \\
&= \log\frac{x^2(x+2)}{(x-3)(x+2)} \\
&= \log\frac{x^2}{x-3}.
\end{aligned}
$$

□

The law of forgetting

Example

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c \log(t+1),$$

where c is a constant that depends on the type of task and t is measured in months.

(1) Solve for P.

(2) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

The law of forgetting

Solution.

(1) $\log P = \log P_0 - c\log(t+1) = \log \dfrac{P_0}{(t+1)^c}$ implies

$$P = \frac{P_0}{(t+1)^c}.$$

(2) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.
In 2 months: $P = \dfrac{90}{3^{0.2}} \approx 72.$
In 1 year: $P = \dfrac{90}{13^{0.2}} \approx 54.$
Your expected scores after 2 months and after 1 year are 72 and 54, respectively.

□

Change of base formula

Proposition

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Change of base formula

Corollary

(1) $\log_a b = \dfrac{1}{\log_b a}$.

(2) $\log_{a^c} b = \dfrac{\log_a b}{c}$.

(3) $a^{\log_b c} = c^{\log_b a}$.

Change of base formula

The change of base formula is quite useful when evaluating logarithms using calculators:

Example

Evaluate each logarithm:

(1) $\log_8 5$.

(2) $\log_9 20$.

(3) $\log_4 3 \cdot \log_9 64$.

(4) $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 2$.

(5) $\log_{16} 27 \cdot \log_{81} 32$.

Change of base formula

Solution.

(1) $\log_8 5 = \dfrac{\log 5}{\log 8} \approx 0.77398.$

(2) $\log_9 20 = \dfrac{\ln 20}{\ln 9} \approx 1.36342.$

(3) $\log_2 3 \cdot \log_9 8 = \dfrac{\log 3 \cdot \log 8}{\log 2 \cdot \log 9} = \log_9 3 \cdot \log_2 8 = \dfrac{1}{2} \cdot 3 = \dfrac{3}{2}.$

(4) $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 2 = \log_3 3 \cdot \log_4 4 \cdot \log_5 5 \cdot \log_2 2 = 1.$

(5) $\log_{16} 27 \cdot \log_{81} 32 = \dfrac{3}{4} \log_2 3 \cdot \dfrac{5}{4} \log_3 2 = \dfrac{15}{16}.$

□