Fundamentals of Mathematics (MATH 1510)

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Outline

Exponential equations

An exponential equation is one in which the variable occurs in the exponent.

Some exponential equations can be solved by using the fact that exponential functions are one-to-one, i.e.,

$$a^x = a^y \iff x = y.$$

Exponential equations

Example

Solve the exponential equation

$$3^{x+2} = 7.$$

Exponential equations

Solution.

$$
\begin{aligned}
3^{x+2} &= 7, \\
\log_3 3^{x+2} &= \log_3 7, \\
x + 2 &= \log_3 7, \\
x &= \log_3 7 - 2.
\end{aligned}
$$

□

Exponential equations

In this example,

$$x = \log_3 7 - 2$$

is the exact solution of the equation $3^{x+2} = 7$.

Using calculators, one could find an approximation to the exact solution by the change of base formula:

$$x = \log_3 7 - 2 = \frac{\ln 7}{\ln 3} - 2 \approx -0.228756.$$

Exponential equations

Example

Solve the equation

$$e^{2x} - e^{x} - 6 = 0.$$

Exponential equations

Solution.

$$\begin{aligned} e^{2x} - e^x - 6 &= 0, \\ (e^x - 3)(e^x + 2) &= 0, \\ e^x = 3 \quad \text{or} \quad e^x &= -2. \end{aligned}$$

Since $e^x > 0$, one could only have $e^x = 3$ and consequently

$$x = \ln 3.$$

□

Logarithmic equations

A logarithmic equation is one in which a logarithm of the variable occurs.

Some exponential equations can be solved by using the fact that logarithmic functions are one-to-one, i.e.,

$$\log_a x = \log_a y \iff x = y.$$

Logarithmic equations

Example

Solve each equation for x.

(1) $\ln x = 8$.

(2) $\log_2(25 - x) = 3$.

Logarithmic equations

Solution.

(1)

$$\begin{aligned} e^{\ln x} &= e^8, \\ x &= e^8. \end{aligned}$$

(2)

$$\begin{aligned} \log_2(25 - x) &= 3, \\ 2^{\log_2(25-x)} &= 2^3, \\ 25 - x &= 8, \\ x &= 17. \end{aligned}$$

□

Compound interest

If a principal P is invested at an annual interest rate r for a period of t years, then the amount A of the investment is given by

- $A(t) = P(1 + rt)$ (simple interest),
- $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ (interest compounded n times per year),
- $A(t) = Pe^{rt}$ (interest compounded continuously).

Compound interest

Example

A principal P is invested at an interest rate r per year. Find the time required for the money to double if the interest is

(1) simple,

(2) compounded semiannually,

(3) compounded continuously.

Compound interest

Solution.

Suppose that the money will be double after t years.

(1) Simple interest:

$$\begin{aligned} P(1+rt) &= 2P, \\ 1+rt &= 2, \\ t &= \frac{1}{r}. \end{aligned}$$

Compound interest

(2) Interest compounded semiannually:

$$
\begin{aligned}
P\left(1+\frac{r}{2}\right)^{2t} &= 2P, \\
\left(1+\frac{r}{2}\right)^{2t} &= 2, \\
\log_{1+\frac{r}{2}}\left(1+\frac{r}{2}\right)^{2t} &= \log_{1+\frac{r}{2}} 2, \\
2t &= \log_{1+\frac{r}{2}} 2, \\
t &= \frac{1}{2}\log_{1+\frac{r}{2}} 2.
\end{aligned}
$$

Compound interest

(3) Interest compounded continuously:

$$\begin{aligned} Pe^{rt} &= 2P, \\ e^{rt} &= 2, \\ \ln e^{rt} &= \ln 2, \\ rt &= \ln 2, \\ t &= \frac{\ln 2}{r}. \end{aligned}$$

□

Compound interest

For example, if the annual interest rate is 5%, the number of years t needed to double an investment is:

(1) Simple interest:

$$t = \frac{1}{r} = \frac{1}{0.05} = 20.$$

(2) Interest compounded semiannually:

$$t = \frac{1}{2} \log_{1+\frac{r}{2}} 2 = \frac{\ln 2}{2 \ln 1.025} \approx 14.04.$$

(3) Interest compounded continuously:

$$t = \frac{\ln 2}{r} = \frac{\ln 2}{0.05} \approx 13.86.$$