Fundamentals of Mathematics (MATH 1510)

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Outline

Exponential growth (doubling time)

Suppose a single bacterium divides every hour. Then we can model the bacteria population after t hours by

$$f(t) = 2^t.$$

If we start with 10 of such bacteria, then the population is modeled by

$$f(t) = 10 \cdot 2^t.$$

If these bacteria doubles every 3 hours, then the population is modeled by

$$f(t) = 10 \cdot 2^{\frac{t}{3}}.$$

Exponential growth (doubling time)

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If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$n(t) = n_0 2^{\frac{t}{a}},$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

Exponential growth (doubling time)

Example

Under ideal conditions a certain bacteria population doubles every three hours. Initially there are 1000 bacteria in a colony.

(1) Find a model for the bacteria population after t hours.

(2) How many bacteria are in the colony after 15 hours?

(3) After how many hours will the bacteria count reach $100,000$?

Exponential growth (doubling time)

Solution.

(1) The population at time t is modeled by

$$n(t) = 1000 \cdot 2^{\frac{t}{3}},$$

where t is measured in hours.

(2) After 15 hours the number of bacteria is

$$n(15) = 1000 \cdot 2^{\frac{15}{3}} = 32,000.$$

(3) We set $n(t) = 100,000$ in the model and solve the resulting equation

$$100,000 = 1000 \cdot 2^{\frac{t}{3}}$$

for t.

Exponential growth (doubling time)

$$
\begin{aligned}
100,000 &= 1000 \cdot 2^{\frac{t}{3}}, \\
100 &= 2^{\frac{t}{3}}, \\
\log 100 &= \log 2^{\frac{t}{3}}, \\
2 &= \frac{t}{3} \log 2, \\
t &= \frac{6}{\log 2} \approx 19.93.
\end{aligned}
$$

Hence the bacteria level reaches 100,000 after about 19.93 hours. □

Exponential growth (relative growth rate)

We have used an exponential function with base 2 to model population growth (in terms of the doubling time).

We could also model the same population with an exponential function with base 3 (in terms of the tripling time).

In fact, we can find an exponential model with any base. If we use the base e, we get a population model in terms of the relative growth rate r: the rate of population growth expressed as a proportion of the population at any time.

Exponential growth (relative growth rate)

Exponential growth (relative growth rate)

A population that experiences exponential growth increases according to the model

$$n(t) = n_0 e^{rt},$$

where

$$\begin{aligned} n(t) &= \text{population at time } t, \\ n_0 &= \text{initial size of the population}, \\ r &= \text{relative rate of growth}, \\ t &= \text{time}. \end{aligned}$$

Notice that the formula for population growth is the same as that for continuously compounded interest.

Exponential growth (relative growth rate)

Example

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

(1) Find a function that models the number of bacteria after t hours.

(2) What is the estimated count after 10 hours?

(3) After how many hours will the bacteria count reach $80,000$?

Exponential growth (relative growth rate)

Solution.

(1) The number of bacteria after t hours is modeled by

$$n(t) = 500e^{0.4t}.$$

(2) The bacterium count after 10 hours is

$$n(10) = 500e^{4} \approx 27,300.$$

(3) We set $n(t) = 80,000$ and solve the resulting exponential equation

$$80,000 = 500e^{0.4t}$$

for t.

Exponential growth (relative growth rate)

$$
\begin{aligned}
80,000 &= 500e^{0.4t},\\
160 &= e^{0.4t},\\
\ln 160 &= 0.4t,\\
t &= \frac{\ln 160}{0.4} \approx 12.68.
\end{aligned}
$$

Hence the bacteria level reaches 80,000 after about 12.68 hours. □

Radioactive decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is proportional to the mass of the substance.

This is analogous to population growth except that the mass decreases. Physicists express the rate of decay in terms of half-life, the time it takes for a sample of the substance to decay to half its original mass.

For example, the half-life of radium-226 is 1600 years, so a 100-g sample decays to 50 g in 1600 years, then to 25 g in 3200 years, and so on.

Radioactive decay

In general, for a radioactive substance with mass m_0 and half-life h, the amount remaining at time t is modeled by

$$m(t) = m_0 2^{-\frac{t}{h}},$$

where h and t are measured in the same time units (minutes, hours, days, years, and so on).

The model can be expressed in the form

$$m(t) = m_0 e^{-\frac{t \ln 2}{h}}$$

with base e.

Radioactive decay

Radioactive decay model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt},$$

where $r = \frac{\ln 2}{h}$ is the relative decay rate.

Radioactive decay

Example

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

(1) Find a function $m(t) = m_0 2^{-\frac{t}{h}}$ that models the mass remaining after t days.

(2) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

(3) Find the mass remaining after one year.

(4) How long will it take for the sample to decay to a mass of 200 mg?

Radioactive decay

Solution.

(1) The amount remaining after t days is

$$m(t) = 300 \cdot 2^{-\frac{t}{140}}.$$

(2) Since the relative decay rate $r = \dfrac{\ln 2}{140} \approx 0.00495$, the amount remaining after t days is

$$m(t) = 300 \cdot e^{-0.00495t}.$$

(3) The mass remaining after one year is

$$m(365) = 300 \cdot e^{-0.00495 \cdot 365} \approx 49.256.$$

Thus approximately 49.256 mg of ^{210}Po remains after 1 year.

Radioactive decay

(4) We set $m(t) = 200$ and solve the resulting exponential equation for t:

$$\begin{aligned} 200 &= 300 \cdot e^{-0.00495t}, \\ \frac{2}{3} &= e^{-0.00495t}, \\ \ln\frac{2}{3} &= -0.00495t, \\ t &= -\frac{\ln\frac{2}{3}}{0.00495} \approx 81.9. \end{aligned}$$

Hence the time required for the sample to decay to 200 mg is about 81.9 days.

$\square$