Fundamentals of Mathematics (MATH 1510)

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Outline

1. Sequences and Summation Notation
2. Arithmetic Sequences

Sequences

Definition

A sequence is a function $a : \mathbb{N}^+ \longrightarrow \mathbb{R}$. The terms of the sequence are the function values, usually denoted by

$$a_n = a(n)$$

for all $n \in \mathbb{N}^+$. So the terms of the sequence are written as

$$a_1, a_2, \ldots, a_n, \ldots,$$

where a_1 is the first term and, in general, a_n is the nth term.

Terms of a sequence

Example

Find the first five terms and the 100th term of the sequence defined by each formula:

(1) $a_n = 2n - 1$.

(2) $c_n = n^2 - 1$.

(3) $t_n = \dfrac{n}{n+1}$.

(4) $r_n = \dfrac{(-1)^n}{2^n}$.

Terms of a sequence

Solution.

(1) $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$, $a_5 = 9$, $a_{100} = 199$.

(2) $c_1 = 0$, $c_2 = 3$, $c_3 = 8$, $c_4 = 15$, $c_5 = 24$, $c_{100} = 9999$.

(3) $t_1 = \frac{1}{2}$, $t_2 = \frac{2}{3}$, $t_3 = \frac{3}{4}$, $t_4 = \frac{4}{5}$, $t_5 = \frac{5}{6}$, $t_{100} = \frac{100}{101}$.

(4) $r_1 = -\frac{1}{2}$, $r_2 = \frac{1}{4}$, $r_3 = -\frac{1}{8}$, $r_4 = \frac{1}{16}$, $r_5 = -\frac{1}{32}$, $r_{100} = \frac{1}{2^{100}}$.

□

Terms of a sequence

Example

Find the nth term of a sequence whose first several terms are given:

(1) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots.$

(2) $-2, 4, -8, 16, -32, \ldots.$

Terms of a sequence

Solution.

(1) $a_n = \dfrac{2n-1}{2n}$.

(2) $a_n = (-1)^n 2^n$.

□

Recursively defined sequences

Some sequences do not have simple defining formulas like those of the preceding example.

The nth term of a sequence may depend on some or all of the terms preceding it. A sequence defined in this way is called recursive.

The Fibonacci sequence

Example (The Fibonacci sequence)

Find the first 11 terms of the sequence defined recursively by $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

for all $n \geq 3$.

The Fibonacci sequence

Solution.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots.$$

□

The Fibonacci sequence and the golden ratio

The limit

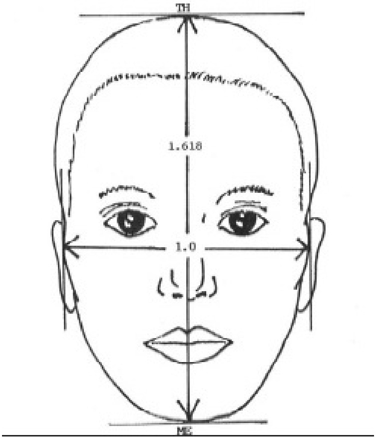
$$\varphi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is known as the golden ratio, which is believed as the key to creating aesthetically pleasing art by artists and architects.

Golden ratio

The Parthenon in Greece

Golden ratio



The partial sum of a sequence

Definition

For a sequence

$$a_1, a_2, \ldots, a_n, \ldots,$$

the nth partial sum is defined as

$$S_n = a_1 + a_2 + \cdots + a_n$$

for all $n \in \mathbb{N}^+$. The sequence

$$S_1, S_2, \ldots, S_n, \ldots$$

is called the sequence of partial sums.

The partial sum of a sequence

Example

Find the first four partial sums and the nth partial sum of the sequence given by $a_n = \dfrac{1}{2^n}$.

The partial sum of a sequence

Solution.

The terms of the sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots.$$

Thus

$$S_1 = \frac{1}{2},$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8},$$
$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

The partial sum of a sequence

In general, the nth partial sum is

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

□

Sigma notation

Given a sequence

$$a_1, a_2, \ldots, a_n, \ldots,$$

we can write the sum of the first n terms using summation notation, or sigma notation as

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The left side of this expression reads as "The sum of a_k from $k = 1$ to $k = n$."

Sigma notation

The letter k is called the index of summation, or the summation variable.

The idea is to replace k in the expression after the sigma by the integers $1, 2, 3, \ldots, n$, and add the resulting expressions, arriving at the right-hand side of the equation.

Sigma notation

Example

Find each sum:

(1) $\sum_{k=1}^{5} k^2$.

(2) $\sum_{j=3}^{5} \frac{1}{j}$.

(3) $\sum_{k=5}^{10} k$.

(4) $\sum_{i=1}^{6} 2$.

Sigma notation

Solution.

(1) $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$

(2) $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}.$

(3) $\sum_{k=5}^{10} k = 5 + 6 + 7 + 8 + 9 + 10 = 45.$

(4) $\sum_{i=1}^{6} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 12.$

$\square$

Properties of sums

Proposition

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for all $n \in \mathbb{N}^+$ and $c \in \mathbb{R}$, the following properties hold:

(1) $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

(2) $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k.$$

(3) $$\sum_{k=1}^{n} ca_k = c\Big(\sum_{k=1}^{n} a_k\Big).$$

Outline

Arithmetic sequences

Definition

An arithmetic sequence is a sequence of the form

$$a, a + d, a + 2d, a + 3d, \dots.$$

The nth term of an arithmetic sequence is given by

$$a_n = a + (n - 1)d,$$

where a is the first term and d is the common difference of the sequence.

Terms of an arithmetic sequence

Example

Find the common difference, the first six terms, the nth term, and the 300th term of the arithmetic sequence

$$13, 7, 1, -5, \ldots .$$

Terms of an arithmetic sequence

Solution.

The common difference $d = 7 - 13 = -6$. The nth term is

$$a_n = 13 - 6(n-1) = 19 - 6n.$$

The first six terms are

$$13, 7, 1, -5, -11, -17.$$

The 300th term is

$$a_{300} = 19 - 6 \cdot 300 = -1781.$$

□

Terms of an arithmetic sequence

Example

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

Terms of an arithmetic sequence

Solution.

Suppose $a_n = a + (n-1)d$, then

$$\begin{cases} a_{11} = a + 10d = 52, \\ a_{19} = a + 18d = 92. \end{cases}$$

Solving this system we get $a = 2$ and $d = 5$. Thus

$$a_n = 2 + 5(n-1) = 5n - 3,$$

and consequently $a_{1000} = 4997$. □

Partial sums of arithmetic sequences

For an arithmetic sequence given by

$$a_n = a + (n-1)d,$$

we want to find the sum of the the first n terms

$$\begin{aligned} S_n &= \sum_{k=1}^{n} [a + (k-1)d] \\ &= a + (a+d) + (a+2d) + \cdots + [a + (n-1)d]. \end{aligned}$$

Partial sums of arithmetic sequences

To this end, note that

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \ldots + [a+(n-1)d], \\ S_n &= [a+(n-1)d] + [a+(n-2)d] + [a+(n-3)d] + \ldots + a. \end{aligned}$$

Thus

$$2Sn = n[a + a + (n-1)d] = n[2a + (n-1)d],$$

and consequently

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a_1 + a_n).$$

Partial sums of arithmetic sequences

Theorem

For the arithmetic sequence given by $a_n = a + (n-1)d$, the nth partial sum

$$S_n = \sum_{k=1}^{n} [a + (k-1)d]$$

is given by either of the following formulas:

- $S_n = \frac{n}{2}[2a + (n-1)d]$.
- $S_n = \frac{n}{2}(a_1 + a_n)$.

Partial sums of arithmetic sequences

Example

Find the sum of the first 50 odd numbers.

Partial sums of arithmetic sequences

Solution.

The odd numbers form an arithmetic sequence with $a = 1$ and $d = 2$, i.e.,

$$a_n = 1 + 2(n - 1) = 2n - 1.$$

So the 50th odd number is

$$a_{50} = 2 \cdot 50 - 1 = 99,$$

and consequently

$$S_{50} = \frac{50}{2}(1 + 99) = 2500.$$

□