Fundamentals of Mathematics (MATH 1510)

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Outline

Geometric sequences

Definition

A geometric sequence is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots.$$

The *n*th term of a geometric sequence is given by

$$a_n = ar^{n-1},$$

where a is the first term and r is the common ratio of the sequence.

Examples of geometric sequences

Example

(1) If $a = 3$ and $r = 2$, then we have the geometric sequence

$$3,\ 3\cdot 2,\ 3\cdot 2^2,\ 3\cdot 2^3,\ 3\cdot 2^4, \ldots,$$

i.e.,

$$3, 6, 12, 24, 48, \ldots.$$

The nth term is

$$a_n = 3\cdot 2^{n-1}.$$

Examples of geometric sequences

(2) The sequence

$$2, -10, 50, -250, 1250, \ldots$$

is a geometric sequence with $a = 2$ and $r = -5$.

When r is negative, the terms of the sequence alternate in sign.

The nth term is

$$a_n = 2 \cdot (-5)^{n-1}.$$

Examples of geometric sequences

(3) The sequence

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

is a geometric sequence with $a = 1$ and $r = \frac{1}{3}$.

The nth term is

$$a_n = \frac{1}{3^{n-1}}.$$

Geometric sequences

If $r > 0$ and $r \neq 1$, then the points of the geometric sequence

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

lie in the graph of the exponential function

$$f(x) = ar^{x-1}.$$

If $a > 0$, then the terms of the geometric sequence ar^{n-1} decrease when $0 < r < 1$ and increase when $r > 1$.

Finding terms of a geometric sequence

Example

Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence

$$5, 15, 45, 135, \ldots.$$

Finding terms of a geometric sequence

Solution.

The first term $a = 5$ and the common ratio $r = \dfrac{45}{15} = 3$. Then

$$a_n = 5 \cdot 3^{n-1}$$

and

$$a_8 = 5 \cdot 3^7 = 5 \cdot 2187 = 10935.$$

□

Partial sums of geometric sequences

For the geometric sequence

$$a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots,$$

the nth partial sum is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}.$$

If $r = 1$, then it is clear that

$$S_n = na.$$

Partial sums of geometric sequences

To find a formula for S_n when $r \neq 1$, note that

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}, \\ rS_n &= \phantom{a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n. \end{aligned}$$

Thus

$$\begin{aligned} S_n - rS_n &= a - ar^n, \\ (1-r)S_n &= a(1-r^n), \\ S_n &= \frac{a(1-r^n)}{1-r}. \end{aligned}$$

Partial sums of geometric sequences

Theorem

For the geometric sequence defined by $a_n = ar^{n-1}$, the nth partial sum

$$S_n = \sum_{k=1}^{n} ar^{k-1}$$

is given by

$$S_n = \begin{cases} na & \text{if } r = 1, \\ \dfrac{a(1-r^n)}{1-r} = \dfrac{a_1 - a_{n+1}}{1-r} & \text{if } r \neq 1. \end{cases}$$

Finding the partial sum of a geometric sequence

Example

Find the following partial sum of a geometric sequence:

$$1 + 4 + 16 + \cdots + 4096.$$

Finding the partial sum of a geometric sequence

Solution.

For this geometric sequence $a = 1$ and $r = 4$, thus $a^n = 4^{n-1}$.

Since $4^6 = 4096$, it follows that

$$S_7 = \frac{1 - 4^7}{1 - 4} = \frac{1 - 16384}{1 - 4} = \frac{16383}{3} = 5461.$$

□

Infinite series

An expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \dots$$

is called an infinite series.

As n gets larger and larger, we are adding more and more of the terms of this series. Intuitively, as n gets larger, the partial sum S_n gets closer to the sum of the series.

Infinite series

Consider the geometric sequence defined by $a_n = \dfrac{1}{2^n}$. The nth partial sum

$$S_n = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}.$$

Since $\dfrac{1}{2^n} \to 0$ as $n \to \infty$, $S_n \to 1$ as $n \to \infty$; that is,

$$\lim_{n\to\infty} S_n = 1.$$

Infinite series

In general, if the nth partial sum S_n of a series $\sum_{k=1}^{\infty} a_k$ gets close to a finite number S as n gets large, i.e.,

$$\lim_{n\to\infty} S_n = S,$$

we say that the infinite series converges (or is convergent). The number S is called the sum of the infinite series.

If an infinite series does not converge, we say that the series diverges (or is divergent).

Infinite geometric series

An infinite geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \ldots.$$

Since the nth partial sum of such as series is

$$S_n = \frac{a(1-r^n)}{1-r}$$

whenever $r \neq 1$, it follows that

$$S_n \to \frac{a}{1-r} \quad \text{as} \quad n \to \infty$$

whenever $|r| < 1$, and consequently $\dfrac{a}{r-1}$ is the sum of this infinite geometric series.

Infinite geometric series

Theorem

If $|r| < 1$*, then the infinite geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \dots$$

converges and has the sum

$$S = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the series diverges.*

Infinite geometric series

Example

Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(1) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \dots.$

(2) $1 + \frac{7}{5} + \frac{49}{25} + \frac{343}{125} + \dots.$

Infinite geometric series

Solution.

(1) This is an infinite geometric series with $a = 2$ and $r = \frac{1}{5}$. Since $|r| < 1$, the series converges and the sum is

$$S = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2}.$$

(2) This is an infinite geometric series with $a = 1$ and $r = \frac{7}{5}$. Since $|r| > 1$, the series diverges.

□

Outline

The amount of an annuity

An annuity is a sum of money that is paid in regular equal payments. Although the word annuity suggests annual payments, they can be made semiannually, quarterly, monthly, or at some other regular interval.

Payments are usually made at the end of the payment interval. The amount of an annuity is the sum of all the individual payments from the time of the first payment until the last payment is made, together with all the interest.

We denote this sum by A_f (the subscript f here is used to denote final amount).

The amount of an annuity

Example

An investor deposits \$400 every December 15 and June 15 for 10 years in an account that earns interest at the rate of 8% per year, compounded semiannually. How much will be in the account immediately after the last payment?

The amount of an annuity

Solution.

$$\begin{aligned} A_f &= 400 \cdot 1.04^{19} + 400 \cdot 1.04^{18} + \cdots + 400 \cdot 1.04 + 400 \\ &= 400 \sum_{k=1}^{20} 1.04^{k-1} \\ &= 400 \cdot \frac{1 - 1.04^{20}}{1 - 1.04} \\ &\approx 11,911.23. \end{aligned}$$

Thus the amount in the account after the last payment is \$11,911.23. □

The amount of an annuity

In general, the regular annuity payment is called the periodic rent and is denoted by R.

We also let i denote the interest rate per time period and let n denote the number of payments. If the time period in which interest is compounded is equal to the time between payments, then the amount A_f of an annuity is

$$\begin{aligned} A_f &= R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots + R(1+i) + R \\ &= R\frac{1-(1+i)^n}{1-(1+i)} = R\frac{1-(1+i)^n}{-i} = R\frac{(1+i)^n - 1}{i}. \end{aligned}$$

The amount of an annuity

Theorem

The amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period is given by

$$A_f = R\frac{(1+i)^n - 1}{i}.$$

The present value

If you were to receive $10,000 five years from now, it would be worth much less than if you got $10,000 right now.

What smaller amount would you be willing to accept now instead of receiving $10,000 in 5 years? The amount that we are looking for here is called the discounted value or present value.

The present value

If the interest rate is 8% per year, compounded quarterly, then the interest per time period is $i = \dfrac{0.08}{4} = 0.02$, and there are $4 \cdot 5 = 20$ time periods.

Let A_p denote the present value, then

$$10,000 = A_p(1+i)^n = A_p \cdot 1.02^{20}$$

and consequently

$$A_p = 10,000 \cdot 1.02^{-20} \approx 6729.71.$$

That is, in this situation, receiving \$6729.71 at present is equivalent to receiving \$10,000 in 5 years.

The present value

In general, if an amount A_f is to be paid in a lump sum n time periods from now and the interest rate per time period is i, then its present value A_p is given by

$$A_p = A_f(1 + i)^{-n}.$$

The present value of an annuity

Similarly, the present value of an annuity is the amount A_p that must be invested now at the interest rate i per time period to provide n payments, each of amount R.

It is clear that A_p is the sum of the present values of each individual payment. However, an easier way of finding A_p is to note that A_p is the present value of A_f; that is,

$$A_p = A_f(1+i)^{-n} = R\frac{(1+i)^n - 1}{i}(1+i)^{-n} = R\frac{1-(1+i)^{-n}}{i}.$$

The present value of an annuity

Theorem

The present value A_p *of an annuity consisting of* n *regular equal payments of size* R *and interest rate* i *per time period is given by*

$$A_p = R\frac{1-(1+i)^{-n}}{i}.$$

The present value of an annuity

Example

A person wins $10,000,000 in the California lottery, and the amount is paid in yearly installments of half a million dollars each for 20 years. What is the present value of his winnings? Assume that he can earn 10% interest, compounded annually.

The present value of an annuity

Solution.

Since the amount won is paid as an annuity, where $i = 0.1$, $R = 500,000$ and $n = 20$, thus

$$A_p = 500,000 \frac{1 - 1.1^{-20}}{0.1} \approx 4,256,781.86.$$

This means that the winner really won only $4,256,781.86 if it were paid immediately. □

Installment buying

Example

A student wishes to buy a car. She can afford to pay \$200 per month but has no money for a down payment. If she can make these payments for 4 years and the interest rate is 12%, what purchase price can she afford?

Installment buying

Solution.

The payments that the student makes constitute an annuity whose present value is the price of the car (which is also the amount of the loan, in this case), where $i = \frac{0.12}{12} = 0.01$, $R = 200$ and $n = 12 \cdot 4 = 48$. Thus

$$A_p = 200\frac{1 - 1.01^{-48}}{0.01} \approx 7594.79.$$

This means the student can buy a car priced at \$7594.79. □