Fundamentals of Mathematics (MATH 1510)

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Outline

1 Mathematical Induction

Conjecture and proof

Let us start from a simple example. We add more and more of the odd numbers as follows:

$$
\begin{aligned}
1 &= 1, \\
1+3 &= 4, \\
1+3+5 &= 9, \\
1+3+5+7 &= 16, \\
1+3+5+7+9 &= 25.
\end{aligned}
$$

One may observe that the numbers on the right hand sides of these equations are all perfect squares; that is,

- For $1 \leq n \leq 5$, the sum of the first n odd numbers is n^2.

Then it is natural to ask: is the above statement true for every positive integer n?

Conjecture and proof

It is not difficult to try more numbers and find that the pattern persists for the first 6, 7, 8, 9, and 10 odd numbers. At this point we feel fairly confident that this is always true, so we make a conjecture:

- The sum of the first n odd numbers is n^2.

Since the nth odd number is $2n - 1$, we can write this statement more precisely as

- $\sum_{i=1}^{n}(2i - 1) = n^2$.

But this is still a conjecture.

Conjecture and proof

We cannot conclude by checking a finite number of cases that a property is true for all $n \in \mathbb{N}^+$ since there are infinitely many positive integers.

A proof is a clear argument that demonstrates the truth of a statement beyond doubt.

Conjecture and proof

Conjecture

For every positive integers n,

$$\sum_{i=1}^{n}(2i-1)=n^2.$$

Conjecture and proof

Proof.

The odd numbers constitute an arithmetic sequence given by $a_n = 2n - 1$. Using the formula of partial sums of an arithmetic sequence, one concludes that

$$S_n = n \cdot \frac{1 + (2n - 1)}{2} = n^2.$$

Hence $\sum_{i=1}^{n}(2i - 1) = n^2$. □

Mathematical induction

Now we introduce a special kind of proof called mathematical induction, which is useful for statements involving natural numbers n.

For example, let $P(n)$ denote the following statement:

- $P(n)$: The sum of the first n odd numbers is n^2.

Since this statement involves all positive integers, it contains infinitely many statements which will be called $P(1)$, $P(2)$, $P(3)$, . . . :

- $P(1)$: The sum of the first 1 odd number is 1^2.
- $P(2)$: The sum of the first 2 odd numbers is 2^2.
- $P(3)$: The sum of the first 3 odd numbers is 3^2.
- . . . .

Mathematical induction

Theorem (Principle of mathematical induction)

For each positive integer n, let $P(n)$ denote a statement depending on n. Suppose that the following two conditions are satisfied:

1. *$P(1)$ is true;*
2. *for every positive integer k, if $P(k)$ is true then $P(k+1)$ is true.*

Then $P(n)$ is true for all positive integers n.

Mathematical induction

To apply this principle, there are two steps:
Step 1. Prove that $P(1)$ is true.
Step 2. Assume that $P(k)$ is true, and use this assumption to prove that $P(k+1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that if $P(k)$ is true, then $P(k+1)$ is also true. The assumption that $P(k)$ is true is called the induction hypothesis.

Proofs by mathematical induction

Proposition

For all positive integers n,

$$\sum_{i=1}^{n}(2i-1)=n^2.$$

Proofs by mathematical induction

Proof.

Let $P(n)$ denote the statement $\sum_{i=1}^{n}(2i-1) = n^2$.

Step 1. $P(1)$ is true since $1 = 1^2$.

Step 2. Suppose $P(k)$ is true, i.e., $\sum_{i=1}^{k}(2i-1) = k^2$. We show that $P(k+1)$ is true, i.e., $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$.

Indeed,

$$\sum_{i=1}^{k+1}(2i-1) = \left[\sum_{i=1}^{k}(2i-1)\right] + (2k+1) = k^2 + 2k + 1 = (k+1)^2.$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof. $\square$

Sum of powers

Proposition

(1) $\sum_{i=1}^{n} 1 = n.$

(2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

(3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

(4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$

Sum of powers

Proof.

(1) Let $P(n)$ denote the statement $\sum_{i=1}^{n} 1 = n$.

Step 1. $P(1)$ is true since $1 = 1$.

Step 2. Suppose $P(k)$ is true, i.e., $\sum_{i=1}^{k} 1 = k$. We show that $P(k+1)$ is true, i.e., $\sum_{i=1}^{k+1} 1 = k+1$. Indeed,

$$\sum_{i=1}^{k+1} 1 = \left[\sum_{i=1}^{k} 1\right] + 1 = k + 1.$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof.

Sum of powers

(2) Let $P(n)$ denote the statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Step 1. $P(1)$ is true since $1 = \frac{1(1+1)}{2}$.

Step 2. Suppose $P(k)$ is true, i.e.,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

We show that $P(k+1)$ is true, i.e.,

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Sum of powers

Indeed,

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left[\sum_{i=1}^{k} i\right] + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof.

Sum of powers

(3) Let $P(n)$ denote the statement
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1. $P(1)$ is true since $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$.

Step 2. Suppose $P(k)$ is true, i.e.,

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

We show that $P(k+1)$ is true, i.e.,

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Sum of powers

Indeed,

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \left[\sum_{i=1}^{k} i^2\right] + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof.

Sum of powers

(4) Let $P(n)$ denote the statement $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Step 1. $P(1)$ is true since $1^3 = \frac{1^2(1+1)^2}{4}$.

Step 2. Suppose $P(k)$ is true, i.e.,

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

We show that $P(k+1)$ is true, i.e.,

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}.$$

Sum of powers

Indeed,

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \left[\sum_{i=1}^{k} i^3\right] + (k+1)^3\\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3\\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\ &= \frac{(k+1)^2(k^2+4k+4)}{4}\\ &= \frac{(k+1)^2(k+2)^2}{4}.\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof. □

Proofs by mathematical induction

Example

Prove that $4n < 2^n$ for all $n \geq 5$.

Proofs by mathematical induction

Proof.

Let $P(n)$ denote the statement $4n < 2^n$.
Step 1. $P(5)$ is true since $4 \cdot 5 = 20 < 32 = 2^5$.
Step 2. Suppose $P(k)$ is true, i.e., $4k < 2^k$, where $k \geq 5$.
We show that $P(k+1)$ is true, i.e., $4(k+1) < 2^{k+1}$. Indeed,

$$4(k+1) = 4k + 4 < 4k + 4k < 2^k + 2^k = 2^{k+1}.$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof. □