Fundamentals of Mathematics (MATH 1510)

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March 9-11, 2016

Outline

Binomials

An expression of the form $a + b$ is called a binomial. The purpose of this section is to find a formula that gives the expansion of

$$(a+b)^n$$

for all $n \in \mathbb{N}^+$ and we will prove it using mathematical induction.

Expanding $(a+b)^n$

To find a pattern in the expansion of $(a+b)^n$, we first look at some special cases:

$$\begin{aligned}
(a+b)^1 &= a+b, \\
(a+b)^2 &= a^2+2ab+b^2, \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3, \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4, \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5, \\
&\vdots
\end{aligned}$$

Expanding $(a+b)^n$

We notice that the exponents of a decrease and the exponents of b increase:

$$(a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5.$$

The coefficients in the expansions constitute a Pascal's triangle:

$(a+b)^0$						1					
$(a+b)^1$					1		1				
$(a+b)^2$				1		2		1			
$(a+b)^3$			1		3		3		1		
$(a+b)^4$		1		4		6		4		1	
$(a+b)^5$	1		5		10		10		5		1

Expanding $(a + b)^n$

Example

Find the expansion of $(a + b)^7$ using Pascal's triangle.

Expanding $(a+b)^n$

Solution.

Find the seventh row of Pascal's triangle:

$(a+b)^5$			1		5		10		10		5		1		
$(a+b)^6$		1		6		15		20		15		6		1	
$(a+b)^7$	1		7		21		35		35		21		7		1

Thus

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

$\square$

The binomial coefficients

Although Pascal's triangle is useful in finding the binomial expansion for reasonably small values of n, it is not practical for finding $(a+b)^n$ for large values of n.

The reason is that the method we use for finding the successive rows of Pascal's triangle is recursive. Thus to find the 100th row of this triangle, we must first find the preceding 99 rows.

n factorial

The product of the first n natural numbers is denoted by $n!$, called *n factorial*, i.e.,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n.$$

We also define

$$0! = 1$$

which makes many formulas involving factorials shorter and easier to write.

The binomial coefficients

Definition

For $n, r \in \mathbb{N}$ with $r \leq n$, the binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

The binomial coefficients

The binomial coefficients also arises in many areas of mathematics other than algebra, especially in combinatorics.

$\binom{n}{r}$ often reads as "n choose r", because there are $\binom{n}{r}$ ways to choose r elements, disregarding their order, from a set of n elements.

The binomial coefficients

Proposition

$$\binom{n}{r} = \binom{n}{n-r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}.$$

The binomial coefficients

Example

(1) $\binom{9}{4} = \binom{9}{5} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126.$

(2) $\binom{10}{3} = \binom{10}{7} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$

The binomial coefficients

To see the connection between the binomial coefficients and the binomial expansion of $(a+b)^n$, note that

$$\binom{5}{0}=1, \binom{5}{1}=5, \binom{5}{2}=10, \binom{5}{3}=10, \binom{5}{4}=5, \binom{5}{5}=1.$$

These are precisely the entries in the fifth row of Pascal's triangle.

The binomial coefficients

In fact, Pascal's triangle can be expressed as

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
\binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & . & . & . & . & & \binom{n}{n-1} & & \binom{n}{n}
\end{array}
$$

The binomial coefficients

To demonstrate that this pattern holds, we need to show that any entry in this version of Pascal's triangle is the sum of the two entries diagonally above it; that is,

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

for all $n, r \in \mathbb{N}$ with $r \leq n$.

The binomial coefficients

Indeed,

$$\begin{aligned}\binom{n+1}{r} &= \frac{(n+1)!}{r!(n+1-r)!} \\ &= \frac{n![(n+1-r)+r]}{r!(n+1-r)!} \\ &= \frac{n!(n+1-r)}{r!(n+1-r)!} + \frac{n!r}{r!(n+1-r)!} \\ &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)![n-(r-1)]!} \\ &= \binom{n}{r} + \binom{n}{r-1}.\end{aligned}$$

The binomial theorem

We are now ready to state the Binomial Theorem:

Theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$

The binomial theorem

Proof.

Let $P(n)$ denote the statement $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$.

Step 1. $P(1)$ is true since $(a+b)^1 = \binom{1}{0} a^1 + \binom{1}{1} b^1$.

Step 2. Suppose $P(k)$ is true, i.e.,

$$(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} b^r.$$

We show that $P(k+1)$ is true, i.e.,

$$(a+b)^{k+1} = \sum_{r=0}^{k+1} \binom{k+1}{r} a^{k+1-r} b^r.$$

The binomial theorem

Indeed,

$$
\begin{aligned}
(a+b)^{k+1} &= (a+b)\cdot(a+b)^k \\
&= (a+b)\left[\sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^r\right] \\
&= a\left[\sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^r\right] + b\left[\sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^r\right] \\
&= \sum_{r=0}^{k}\binom{k}{r}a^{k+1-r}b^r + \sum_{r=0}^{k}\binom{k}{r}a^{k-r}b^{r+1} \\
&= a^{k+1} + \left[\sum_{r=1}^{k}\binom{k}{r}a^{k+1-r}b^r + \sum_{r=0}^{k-1}\binom{k}{r}a^{k-r}b^{r+1}\right] + b^{k+1}
\end{aligned}
$$

The binomial theorem

$$= a^{k+1} + \left[\sum_{r=1}^{k} \binom{k}{r} a^{k+1-r} b^r + \sum_{r=1}^{k} \binom{k}{r-1} a^{k+1-r} b^r\right] + b^{k+1}$$

$$= a^{k+1} + \left[\sum_{r=1}^{k} \left[\binom{k}{r} + \binom{k}{r-1}\right] a^{k+1-r} b^r\right] + b^{k+1}$$

$$= a^{k+1} + \left[\sum_{r=1}^{k} \binom{k+1}{r} a^{k+1-r} b^r\right] + b^{k+1}$$

$$= \sum_{r=0}^{k+1} \binom{k+1}{r} a^{k+1-r} b^r.$$

Thus $P(k+1)$ follows from $P(k)$, completing the proof. □

The binomial theorem

Example

Use the binomial theorem to expand $(x + y)^4$.

The binomial theorem

Solution.

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

□

The binomial theorem

Example

Find the term that contains x^5 in the expansion of

$$(2x + y)^{20}.$$

The binomial theorem

Solution.

This term is

$$\binom{20}{5}(2x)^5y^{15} = \frac{20\cdot 19\cdot 18\cdot 17\cdot 16}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot 32x^5y^{15}$$
$$= 496,128x^5y^{15}.$$

□