Fundamentals of Mathematics (MATH 1510)

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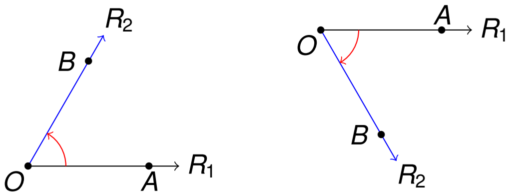
Outline

Angles

An angle $\angle AOB$ consists of two rays R_1 and R_2 with a common vertex O.



Positive angle

Negative angle

We often interpret an angle as a rotation of the ray R_1 (called the initial side) onto R_2 (called the terminal side). If the rotation is counterclockwise, the angle is considered positive, and if the rotation is clockwise, the angle is considered negative.

Angle measure

The measure of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. One unit of measurement for angles is the degree:

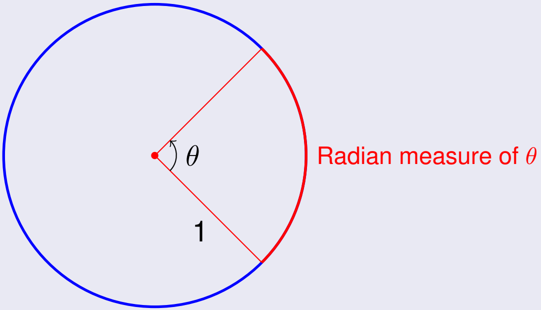
- An angle of measure 1 degree, written as 1°, is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution.

In calculus and other branches of mathematics a more natural method of measuring angles is used: radian measure.

Angle measure

Definition

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in radians (abbreviated rad) is the length of the arc that subtends the angle:



Angle measure

The circumference of the circle of radius 1 is 2π, so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure $\frac{\pi}{2}$ rad.

Similarly, an angle that is subtended by an arc of length 2 along the unit circle has radian measure 2.

Angle measure

From the definition one easily knows the relationship between degrees and radians:

- $180^\circ = \pi$ rad;
- 1 rad $= \left(\frac{180}{\pi}\right)^\circ$;
- $1^\circ = \frac{\pi}{180}$ rad.

For example,

- $60^\circ = \frac{\pi}{3}$ rad,
- $\frac{\pi}{6}$ rad $= 30^\circ$.

Angle measure

We often use a phrase such as “a 30° angle” to mean an angle whose measure is 30°.

Also, for an angle θ we write $\theta = 30°$ or $\theta = \frac{\pi}{6}$ to mean the measure of θ is 30° or $\frac{\pi}{6}$ rad.

When no unit is given, the angle is always assumed to be measured in radians.

Angles in standard position

An angle is in standard position if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis.

Two angles in standard position are coterminal if their sides coincide.

Coterminal angles

Example

(1) Find angles that are coterminal with the angle $\theta = 30^\circ$ in standard position.

(2) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

Coterminal angles

Solution.

(1) $(30 + 360k)^\circ$, where k is any integer.

(2) $\frac{\pi}{3} + 2k\pi$, where k is any integer.

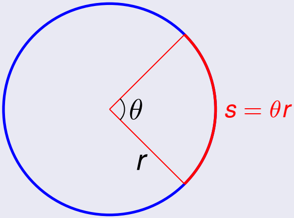
$\square$

Length of a circular arc

Proposition

In a circle of radius r the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta.$$



Length of a circular arc

Proof.

The circumference of a circle of radius r is $2\pi r$. Hence the length s of an arc that subtends a central angle of θ radians is

$$s = 2\pi r \cdot \frac{\theta}{2\pi} = r\theta.$$

□

Length of a circular arc

Therefore, the radian measure of

$$\theta = \frac{s}{r}$$

is the number of “radiuses” that can fit in the arc that subtends θ; hence the term radian.

Length of a circular arc

Example

(1) Find the length of an arc of a circle with radius 10 m that subtends a central angle of 30°.

(2) A central angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ.

Length of a circular arc

Solution.

(1) The angle $\theta = 30^\circ = \dfrac{\pi}{6}$, and thus the length of the arc is

$$s = r\theta = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3}.$$

(2) $\theta = \dfrac{s}{r} = \dfrac{6}{4} = \dfrac{3}{2}$.

□

Area of a circular sector

Proposition

In a circle of radius r the area A of a sector with a central angle θ radians is

$$A = \frac{1}{2} r^2 \theta.$$

Area of a circular sector

Proof.

The area of a circle of radius r is πr^2. Hence the area A of a sector with a central angle θ radians is

$$A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

□

Area of a circular sector

Example

Find the area of a sector of a circle with central angle 60° if the radius of the circle is 3 m.

Area of a circular sector

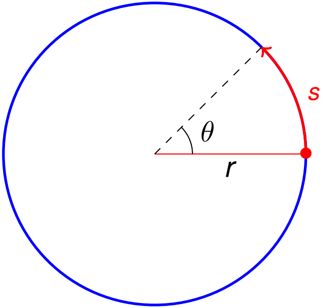
Solution.

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot 3^2 \cdot \frac{\pi}{3} = \frac{3\pi}{2} \mathrm{m}^2.$$

□

Circular motion

Suppose a point moves along a circle as shown below:



There are two ways to describe the motion of the point: linear speed and angular speed.

Circular motion

Linear speed and angular speed

Suppose a point moves along a circle of radius r and the ray from the center of the circle to the point traverses θ radians in time t. Let $s = r\theta$ be the distance the point travels in time t. Then

- the linear speed is given by $v = \dfrac{s}{t}$,
- the angular speed is given by $\omega = \dfrac{\theta}{t}$.

Circular motion

From the definition one soon knows the relationship between linear speed and angular speed:

Proposition

If a point moves along a circle of radius r with angular speed ω, then its linear speed v is given by

$$v = r\omega.$$

Circular motion

Example

A boy rotates a stone in a 3-ft-long sling at the rate of 15 revolutions every 10 seconds. Find the angular and linear velocities of the stone.

Circular motion

Solution.

In 10 s the angle θ changes by $15 \cdot 2\pi = 30\pi$. So the angular speed is

$$\omega = \frac{\theta}{t} = \frac{30\pi}{10} = 3\pi \text{ rad/s}.$$

The linear speed is

$$v = r\omega = 3 \cdot 3\pi = 9\pi \text{ ft/s}.$$

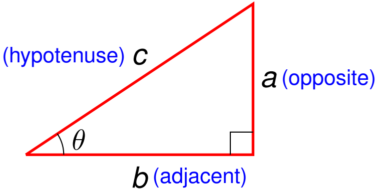
$\square$

Outline

1. Angle Measure
2. Trigonometry of Right Triangles

Trigonometric ratios

Consider a right triangle with θ as one of its acute angles:



The trigonometric ratios are defined as follows:

$$\sin\theta = \frac{a}{c}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{c}{b},$$

$$\cos\theta = \frac{b}{c}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}, \quad \csc\theta = \frac{1}{\sin\theta} = \frac{c}{a}.$$

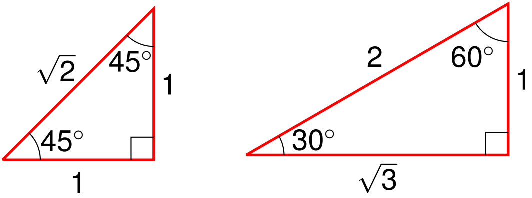
Trigonometric ratios

The symbols we use for these ratios are abbreviations for their full names: sine, cosine, tangent, cotangent, secant, cosecant.

Since any two right triangles with angle θ are similar, these ratios are the same, regardless of the size of the triangle; that is, the trigonometric ratios depend only on the angle θ.

Special triangles

From the special triangles



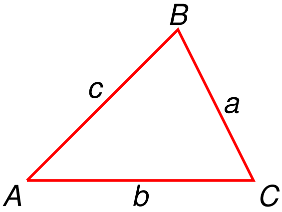
we get the following values of trigonometric ratios:

Special triangles

θ in degrees	θ in radians		$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0		0	1	0	$-$	1	$-$
30°	$\frac{\pi}{6}$		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{\pi}{4}$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
90°	$\frac{\pi}{2}$		1	0	$-$	0	$-$	1

Solving a right triangle

A triangle has six parts: three angles ($\angle A$, $\angle B$, $\angle C$ as below) and three sides (a, b, c as below).

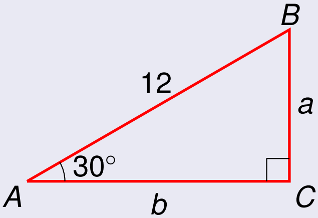


To solve a triangle means to determine all of its parts from the information known about the triangle; that is, to determine the lengths of the three sides and the measures of the three angles.

Solving a right triangle

Example

Solve the triangle:



Solving a right triangle

Solution.

It is clear that $\angle B = 60^\circ$. Since $\sin A = \frac{a}{12}$,

$$a = 12\sin 30^\circ = 12 \cdot \frac{1}{2} = 6.$$

Since $\cos A = \frac{b}{12}$,

$$b = 12\cos 30^\circ = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}.$$

□