Fundamentals of Mathematics (MATH 1510)

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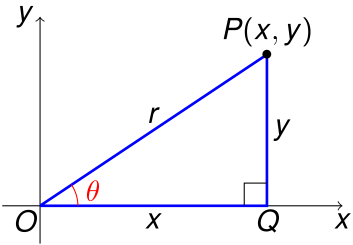
March 21-23, 2016

Outline

Trigonometric functions

Place an acute angle θ in standard position as below and take any point $P(x, y)$ on the terminal side of θ:



Using the Pythagorean Theorem we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$, and thus

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}.$$

The other trigonometric ratios can be found in the same way.

Trigonometric functions

These observations allow us to extend the trigonometric ratios to any angle:

Definition

Let θ be an angle in standard position, and let $P(x, y)$ be a point on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

$$\sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0), \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0),$$

$$\cos\theta = \frac{x}{r}, \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0), \qquad \csc\theta = \frac{r}{y} \quad (y \neq 0).$$

Trigonometric functions

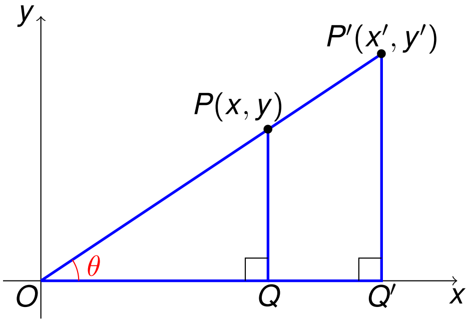
Angles that are coterminal with the coordinate axes are called quadrantal angles. Some trigonometric functions may be undefined for these angles:

- tan and sec are undefined for angles that are coterminal with $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
- cot and csc are undefined for angles that are coterminal with 0 or π.

Trigonometric functions

It is a crucial fact that the values of the trigonometric functions do not depend on the choice of the point $P(x, y)$.

This is because if $P(x', y')$ is any other point on the terminal side, then triangles POQ and $P'OQ'$ are similar:



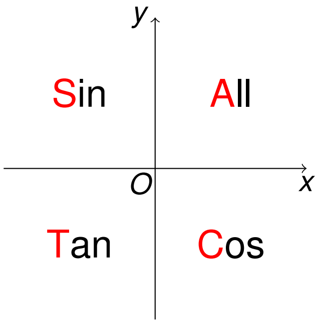
Evaluating trigonometric functions

From the definition we see that the values of the trigonometric functions are all positive if the angle θ has its terminal side in Quadrant I. But their signs may vary in the other quadrants:

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

All Students Take Calculus

All Students Take Calculus is a mnemonic that is used to memorize which trigonometric functions are positive in each quadrant:

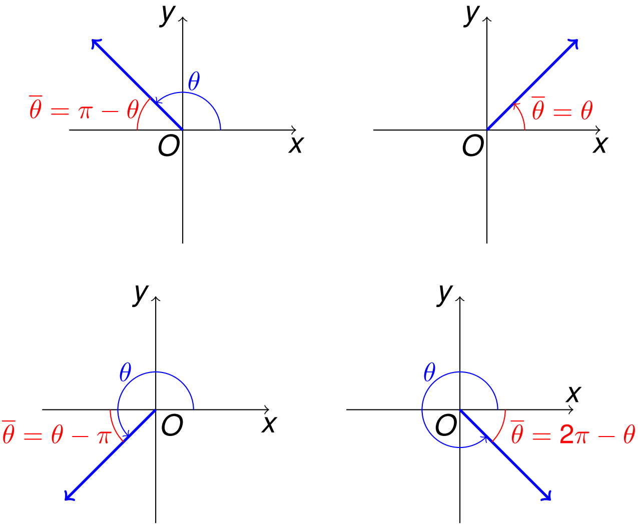


Reference angles

Definition

Let θ be an angle in standard position. The reference angle $\overline{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

Reference angles



Evaluating trigonometric functions

To find the values of the trigonometric functions for any angle θ, we carry out the following steps:

1. Find the reference angle $\overline{\theta}$ associated with the angle θ:
   1. Find the angle $\theta' \in [0, 2\pi)$ that is coterminal with θ;
   2. Find the reference angle of θ' as shown in the above slide, which is exactly the reference angle $\overline{\theta}$ of θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ (or equivalently, θ') lies.
3. The value of the trigonometric function of θ is the same , except possibly for sign, as that of $\overline{\theta}$.

Evaluating trigonometric functions

Example

Find

(1) $\cos 135^\circ$,

(2) $\tan 390^\circ$,

(3) $\csc \dfrac{5\pi}{3}$,

(4) $\cot 870^\circ$,

(5) $\sin \dfrac{16\pi}{3}$,

(6) $\sec\left(-\dfrac{\pi}{4}\right)$.

Evaluating trigonometric functions

Solution.

(1) 135° is in Quadrant II, thus its reference angle is $180^\circ - 135^\circ = 45^\circ$. Since the value of cosine is negative in Quadrant II, we have

$$\cos 135^\circ = -\sin 45^\circ = -\frac{\sqrt{2}}{2}.$$

(2) 390° is coterminal with 30° and they are in Quadrant I. Since the value of tangent is positive in Quadrant I, we have

$$\tan 390^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3}.$$

Evaluating trigonometric functions

(3) $\frac{5\pi}{3}$ is in Quadrant IV, thus its reference angle is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$. Since the value of cosecant is negative in Quadrant IV, we have

$$\csc\frac{5\pi}{3} = -\csc\frac{\pi}{3} = -\frac{2}{\sqrt{3}}.$$

(4) 870° is coterminal with 150° and they are in Quadrant II. Thus the reference angle is $180^\circ - 150^\circ = 30^\circ$. Since the value of cotangent is negative in Quadrant II, we have

$$\cot 870^\circ = \cot 150^\circ = -\cot 30^\circ = -\sqrt{3}.$$

Evaluating trigonometric functions

(5) $\dfrac{16\pi}{3}$ is coterminal with $\dfrac{4\pi}{3}$ and they are in Quadrant III. Thus its reference angle is $\dfrac{4\pi}{3} - \pi = \dfrac{\pi}{3}$. Since the value of sine is negative in Quadrant III, we have

$$\sin\frac{16\pi}{3} = \sin\frac{4\pi}{3} = -\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

(6) $-\dfrac{\pi}{4}$ is in Quadrant IV and its reference angle is $\dfrac{\pi}{4}$. Since the value of secant is positive in Quadrant IV, we have

$$\sec\left(-\frac{\pi}{4}\right) = \sec\frac{\pi}{4} = \sqrt{2}.$$

□

Trigonometric identities

Theorem (Reciprocal identities)

- $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Trigonometric identities

Theorem (Pythagorean identities)

- $\sin^2\theta + \cos^2\theta = 1.$
- $\tan^2\theta + 1 = \sec^2\theta.$
- $1 + \cot^2\theta = \csc^2\theta.$

Trigonometric identities

Example

(1) Express $\sin\theta$ in terms of $\cos\theta$.

(2) Express $\tan\theta$ in terms of $\sin\theta$, where θ is in Quadrant II.

Trigonometric identities

Solution.

(1) Since $\sin^2\theta + \cos^2\theta = 1$, it follows that

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}.$$

If θ is in Quadrant I or II, then $\sin\theta$ is positive, and thus

$$\sin\theta = \sqrt{1-\cos^2\theta}.$$

If θ is in Quadrant III or IV, then $\sin\theta$ is negative, and thus

$$\sin\theta = -\sqrt{1-\cos^2\theta}.$$

Trigonometric identities

(2) Since $\sin^2\theta + \cos^2\theta = 1$ and $\cos\theta$ is negative in Quadrant II, we have

$$\cos\theta = -\sqrt{1 - \sin^2\theta}.$$

Hence

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

□

Areas of triangles

Since the area of a triangle is

$$A = \frac{1}{2} \times \text{base} \times \text{height},$$

it is easy to prove the following theorem:

Theorem

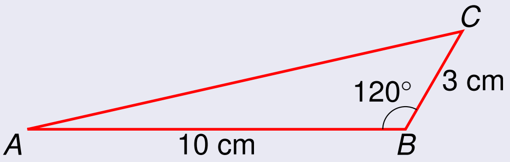
The area A of a triangle with sides of lengths a and b and with included angle θ is

$$A = \frac{1}{2} ab \sin\theta.$$

Areas of triangles

Example

Find the area of the triangle



Areas of triangles

Solution.

The area of the triangle is

$$A = \frac{1}{2} \cdot 10 \cdot 3 \cdot \sin 120^\circ = 15 \sin 60^\circ = \frac{15\sqrt{3}}{2}\ \text{cm}^2.$$

□