FORMAL CONCEPT ANALYSIS ON FUZZY SETS

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Abstract. Formal Concept Analysis (FCA) is a theory on relations between sets, and has already been extended to a theory on fuzzy relations between (crisp) sets. In this note, in the case that $(L, \&)$ is a commutative and divisible complete residuated lattice, FCA is extended further to a theory on fuzzy relations between fuzzy sets. In particular, the fundamental theorem of FCA on fuzzy relations between fuzzy sets is established.

1. Introduction

Formal Concept Analysis (FCA) [5, 4] is a useful tool for qualitative data analysis. A formal context is a triple (X, Y, R), where X, Y are sets and $R \subseteq X \times Y$ is a relation between X and Y. Elements in X are interpreted as objects and those in Y as attributes. Each relation $R \subseteq X \times Y$ induces a pair of operators $R_\uparrow : 2^X \longrightarrow 2^Y$ and $R^\downarrow : 2^Y \longrightarrow 2^X$ as follows:

$$R_\uparrow(U) = \{y \in Y : \forall x \in U, xRy\}$$

and

$$R^\downarrow(V) = \{x \in X : \forall y \in V, xRy\}.$$

This pair of operators is a contravariant Galois connection in the sense that

$$U \subseteq R^\downarrow(V) \iff V \subseteq R_\uparrow(U).$$

This Galois connection plays a fundamental role in formal concept analysis.

A formal concept of a context (X, Y, R) is a pair $(U, V) \in 2^X \times 2^Y$, where U is the *extent* and V the *intent*, such that $U = R^\downarrow(V)$ and $V = R_\uparrow(U)$. The fundamental theorem of formal concept analysis states that the formal concepts of a context form a concept lattice and every complete lattice is the concept lattice of some formal context.

Formal concept analysis is essentially a theory on relations between sets, and has been generalized to a theory on fuzzy relations between crisp sets [2, 3, 1, 6, 12, 10, 14]. Let L be a complete residuated lattice. A fuzzy relation between sets X and Y is a map $R : X \times Y \longrightarrow L$; an L-context is a triple (X, Y, R), where $R : X \times Y \longrightarrow L$ is a fuzzy relation. For an L-context (X, Y, R), Bělohlávek introduced a contravariant fuzzy Galois connection $(R_\uparrow, R^\downarrow)$ between the fuzzy powersets L^X and L^Y, and thus, established formal concept analysis on fuzzy relations between crisp sets.

The aim of this note is to show that it is possible to extend the machinery of FCA further to the realm of fuzzy relations between fuzzy sets (not only fuzzy relations between crisp sets), yielding a theory of formal concept analysis on fuzzy sets.

In this note, we consider the problem in the case that $(L, \&)$ is a commutative and divisible complete residuated lattice. Section II deals with preorders on fuzzy sets; Section III demonstrates that the theory of FCA on relations and fuzzy relations between crisp sets can be extended to a theory on fuzzy relations between fuzzy sets. Due to space limitation, proofs are just sketched here, details will be presented in a paper (in preparation) for an even more general setting.

2. Complete preordered L-subsets

A *commutative complete residuated lattice* [2, 7] is a pair $(L, \&)$, where L is a complete lattice with a bottom element 0 and a top element 1, $\&$ is a binary operation on L such that $(L, \&, 1)$ is a commutative monoid and $a \& \bigvee b_i = \bigvee (a \& b_i)$ for all $a, b_i \in L$.

This work is supported by Natural Science Foundation of China (11071174).

Given a commutative complete residuated lattice $(L, \&)$, the function

$$a\&- : L \longrightarrow L$$

has a right adjoint

$$a \to - : L \longrightarrow L$$

given by $a \to b = \bigvee\{c \in L \mid a\&c \leq b\}$, called the residuation or implication on $(L, \&)$.

Proposition 2.1. [13] *For a commutative complete residuated lattice $(L, \&)$, the following conditions are equivalent:*

1. $\forall a, b \in L,\ a \leq b \Longrightarrow a = b\&(b \to a)$.
2. $\forall a, b, c \in L,\ a, c \leq b \Longrightarrow (b \to a)\&c = a\&(b \to c)$.
3. $\forall a, b \in L,\ a \leq b \Longrightarrow \exists c \in L, a = b\&c$.
4. $\forall a, b \in L,\ a \wedge b = a\&(a \to b)$.

A commutative complete residuated lattice $(L, \&)$ is *divisible* [8] if it satisfies one of the equivalent conditions in Proposition 2.1. Typical examples of such lattices include complete Heyting algebras and complete BL-algebras [7].

From now on, $(L, \&)$ (or, L for short) is always assumed to be a divisible, commutative complete residuated lattice if not otherwise specified.

An *L-subset* (or, a fuzzy set) is a pair (X, λ), where X is a (crisp) set and $\lambda : X \longrightarrow L$ is a map (called the membership function). The value $\lambda(x)$ is interpreted as the membership degree of x in (X, λ). A *fuzzy relation* $R : (X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (Y, \mu)$ between L-subsets is a map $R : X \times Y \longrightarrow L$ such that

$$R(x, y) \leq \lambda(x) \wedge \mu(y) \tag{1}$$

for all $x \in X$ and $y \in Y$. The value $R(x, y)$ is interpreted as the degree of x and y being related, and Equation (1) asserts that the degree of x and y being related can exceed neither the membership degree of x in (X, λ) nor that of y in (Y, μ).

The opposite fuzzy relation of a fuzzy relation $R : (X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (Y, \mu)$ is the fuzzy relation $R^{\rm op} : (Y, \mu) \;{-\!\!\circ\!\!\rightarrow}\; (X, \lambda)$ given by $R^{\rm op}(y, x) = R(x, y)$.

For each L-subset (X, λ), write ${\rm id}_{(X,\lambda)}$ for the fuzzy relation $(X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (X, \lambda)$ given by

$${\rm id}_{(X,\lambda)}(x, y) = \begin{cases} \lambda(x), & x = y; \\ 0, & x \neq y. \end{cases}$$

This fuzzy relation is called the identity fuzzy relation on (X, λ).

Given fuzzy relations $R : (X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (Y, \mu)$ and $S : (Y, \mu) \;{-\!\!\circ\!\!\rightarrow}\; (Z, \nu)$, the composition of S and R [13] is the fuzzy relation

$$S \circ R : (X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (Z, \nu)$$

defined by

$$\begin{aligned} S \circ R(x, z) &= \bigvee_{y \in Y} R(x, y)\&(\mu(y) \to S(y, z)) \\ &= \bigvee_{y \in Y} (\mu(y) \to R(x, y))\&S(y, z). \end{aligned}$$

This formula can be regarded as a many-valued reformulation of the statement that x, z are related if there exists some y in (Y, μ) such that x, y are related and y, z are related.

Proposition 2.2. [13] *Let $R : (X, \lambda) \;{-\!\!\circ\!\!\rightarrow}\; (Y, \mu)$, $S : (Y, \mu) \;{-\!\!\circ\!\!\rightarrow}\; (Z, \nu)$ and $T : (Z, \nu) \;{-\!\!\circ\!\!\rightarrow}\; (W, \gamma)$ be fuzzy relations. Then*

1. $T \circ (S \circ R) = (T \circ S) \circ R$.
2. $R \circ {\rm id}_{(X,\lambda)} = R = {\rm id}_{(Y,\mu)} \circ R$.
3. $(S \circ R)^{\rm op} = R^{\rm op} \circ S^{\rm op}$.

For fuzzy relations $R, S : (X, \lambda) \multimap (Y, \mu)$, let $R \leq S$ if $R(x, y) \leq S(x, y)$ for all $x \in X$ and $y \in Y$. Then all the fuzzy relations from (X, λ) to (Y, μ) form a complete lattice.

Let $R : (X, \lambda) \multimap (X, \lambda)$ be a fuzzy relation on an L-subset (X, λ). Then R is reflexive if $\mathrm{id}_{(X,\lambda)} \leq R$; R is transitive if $R \circ R \leq R$.

Definition 2.3. [13] A *preorder* on an L-subset (X, λ) is a fuzzy relation $P : (X, \lambda) \multimap (X, \lambda)$ that is reflexive and transitive. The pair $((X, \lambda), P)$ is called a preordered L-subset.

Explicitly, a preorder on an L-subset (X, λ) is a map $P : X \times X \longrightarrow L$ such that for all $x, y, z \in X$,

(i) $P(x, y) \leq \lambda(x) \wedge \lambda(y)$,
(ii) $\lambda(x) \leq P(x, x)$,
(iii) $P(x, y)\&(\lambda(y) \to P(y, z)) \leq P(x, z)$.

It follows from (i) and (ii) that $P(x, x) = \lambda(x)$ for all $x \in X$. Therefore, a preordered L-subset can be described as a pair (X, P), where X is a set and $P : X \times X \longrightarrow L$ is a map, such that for all $x, y, z \in X$,

(1) $P(x, y) \leq P(x, x) \wedge P(y, y)$,
(2) $P(x, y)\&(P(y, y) \to P(y, z)) \leq P(x, z)$.

Remark 2.4. From the viewpoint of category theory, a preordered L-subset is a category enriched over a quantaloid constructed from $(L, \&)$ as explained in [9, 13].

A preordered L-subset (X, P) is said to be *separated* if for all $x, y \in X$, $P(x, x) = P(y, y) = P(x, y) = P(y, x)$ implies that $x = y$.

Example 2.5.
(1) For each L-subset (X, λ), the identity fuzzy relation on (X, λ) is a separated preorder on (X, λ), called the discrete preorder on (X, λ). In this note, an L-subset (X, λ) is always assumed to be equipped with the discrete preorder if not otherwise specified.
(2) For each $a \in L$, let $*_a$ be the L-subset $(\{*\}, \lambda)$ given by $\lambda(*) = a$. It is clear that there is exactly one preorder on $*_a$, the discrete one. From now on, $*_a$ is always equipped with this preorder.
(3) If $((X, \lambda), P)$ is a preordered L-subset, then $((X, \lambda), P^{\mathrm{op}})$ is also a preordered L-subset, called the *dual* of $((X, \lambda), P)$.
(4) If $((X, \lambda), P)$ is a preordered L-subset and A is a subset of X. then $P|(A \times A)$ is a preorder on the L-subset $(A, \lambda|A)$. We will write (A, P) for this preordered L-subset.
(5) If $(L, \&) = ([0, +\infty]^{\mathrm{op}}, +)$, then a preordered L-subset is exactly a generalized partial metric space [11, 13].

Definition 2.6. [15] An *order-preserving* map $f : (X, P) \longrightarrow (Y, Q)$ between preordered L-subsets is a map $f : X \longrightarrow Y$ such that $P(x, x) = Q(f(x), f(x))$ and $P(x, x') \leq Q(f(x), f(x'))$ for all $x, x' \in X$. An order-preserving map $f : (X, P) \longrightarrow (Y, Q)$ is said to be *fully faithful* if $P(x, x') = Q(f(x), f(x'))$ for all $x, x' \in X$.

A fully faithful map $f : (X, P) \longrightarrow (Y, Q)$ with (X, P) being separated is necessarily injective. Bijective fully faithful maps are exactly the isomorphisms in the category of preordered L-subsets and order-preserving maps.

Definition 2.7. [15] A pair of order-preserving maps $f : (X, P) \longrightarrow (Y, Q)$ and $g : (Y, Q) \longrightarrow (X, P)$ is called an *adjunction*, written $f \dashv g : (X, P) \rightharpoonup (Y, Q)$, if $Q(f(x), y) = P(x, g(y))$ for all $x \in X$ and $y \in Y$. In this case, f is called a *left adjoint* of g, and g a *right adjoint* of f.

In order to discuss the completeness of preordered L-subsets, we consider first the fuzzy set of lower sets and the fuzzy set of upper sets in a preordered L-subset.

Given a preordered L-subset $((X, \lambda), P)$, the fuzzy set of lower sets in $((X, \lambda), P)$ is the L-subset $(\mathcal{P}(X, P), \Lambda)$ given by the following data:

- $\mathcal{P}(X, P)$ is the set of fuzzy relations

$$\phi : (X, \lambda) \multimap *_{t\phi}$$

satisfying $\phi\circ P\leq\phi$, where $t\phi$ is an element in L.

- The membership function $\Lambda:\mathcal{P}(X,P)\longrightarrow L$ sends each $\phi:(X,\lambda)\relbar\!\!\!\circ\!\!\!\rightarrow *_{t\phi}$ to $t\phi$.

Said differently, an element in $\mathcal{P}(X,P)$ is a map $\phi:X\longrightarrow L$ such that for all $x,y\in X$,

(i) $\phi(x)\leq\lambda(x)\wedge t\phi$,
(ii) $\phi(y)\&(\lambda(y)\rightarrow P(x,y))\leq\phi(x)$.

Since (ii) can be interpreted as the statement that if y is in ϕ and x is smaller than or equal to y then x is in ϕ, so, every element ϕ in $\mathcal{P}(X,P)$ can be regarded as a potential lower fuzzy set in the preordered L-subset (X,P) with $\Lambda(\phi)=t\phi$ being the degree that ϕ is a lower fuzzy set in (X,P).

There is a separated preorder on the L-subset $(\mathcal{P}(X,P),\Lambda)$ [16] given by

$$S_X(\phi,\psi)=t\phi\wedge t\psi\wedge\bigwedge_{x\in X}\Big((t\phi\rightarrow\phi(x))\rightarrow\psi(x)\Big)$$

for $\phi,\psi\in\mathcal{P}(X,P)$.

S_X is intuitively the inclusion order of lower fuzzy sets. The separated preordered L-subset $(\mathcal{P}(X,P),S_X)$ is called the fuzzy powerset of (X,P) and will be abbreviated to $\mathcal{P}(X,P)$ if no confusion arises.

The dual fuzzy powerset $\mathcal{P}^*(X,P)$ of (X,P) is defined to be the dual of $(\mathcal{P}(X,P^{\rm op}),S_X)$, i.e.,

$$\mathcal{P}^*(X,P)=(\mathcal{P}(X,P^{\rm op}),S_X^{\rm op}).$$

Intuitively, $\mathcal{P}^*(X,P)$ is the preordered L-subset of upper fuzzy sets in (X,P) under the converse inclusion order.

Example 2.8 (Fuzzy powerset of a fuzzy set)**.** For each L-subset (X,λ), define the fuzzy powerset, $\mathcal{P}(X,\lambda)$ in symbols, of (X,λ) to be the fuzzy powerset of the discrete preordered L-subset $((X,\lambda),{\rm id}_{(X,\lambda)})$. Dually, the dual fuzzy powerset, $\mathcal{P}^*(X,\lambda)$ in symbols, of (X,λ) is defined to be the dual fuzzy powerset of $((X,\lambda),{\rm id}_{(X,\lambda)})$.

It is easily seen that the dual fuzzy powerset $\mathcal{P}^*(X,\lambda)$ of an L-subset (X,λ) is exactly the dual of the fuzzy powerset of (X,λ), i.e., $\mathcal{P}^*(X,\lambda)=\mathcal{P}(X,\lambda)^{\rm op}$.

Furthermore, if P is a preorder on (X,λ), then $\mathcal{P}(X,P)$ is a subset of $\mathcal{P}(X,\lambda)$ and the order structure on $\mathcal{P}(X,P)$ coincides with that inherited from $\mathcal{P}(X,\lambda)$ as in Example 2.5(4).

Given a preordered L-subset (X,P) and $x\in X$, let $\mathbf{y}(x)$ denote the element ϕ in $\mathcal{P}(X,P)$ given by

$$\phi(y)=P(y,x),\quad t\phi=P(x,x).$$

Dually, let $\mathbf{y}^*(x)$ denote the element ψ in $\mathcal{P}^*(X,P)$ given by

$$\psi(y)=P(x,y),\quad t\psi=P(x,x).$$

The following lemma asserts that both

$$\mathbf{y}:(X,P)\longrightarrow\mathcal{P}(X,P)$$

and

$$\mathbf{y}^*:(X,P)\longrightarrow\mathcal{P}^*(X,P)$$

are fully faithful maps.

Lemma 2.9 (Yoneda)**.**

$$S_X(\mathbf{y}(x),\phi)=\phi(x)$$

and

$$S_X^{\rm op}(\psi,\mathbf{y}'(x))=\psi(x)$$

for all $x\in X$*,* $\phi\in\mathcal{P}(X,P)$ *and* $\psi\in\mathcal{P}^*(X,P)$*.*

In particular, if (X,P) is separated then both $\mathbf{y}$ and $\mathbf{y}^*$ are injective. Therefore, $\mathbf{y}$ and $\mathbf{y}^*$ are called respectively the Yoneda embedding and the co-Yoneda embedding though they need not be injective in general.

Definition 2.10. Suppose $((X,\lambda),P)$ is a preordered L-subset and $\phi\in\mathcal{P}(X,\lambda)$. An element $u\in X$ is a *supremum* of ϕ, denoted by $u=\sup\phi$, if

(1) $P(u,u)=\lambda(u)=t\phi$,
(2) $\forall x\in X,\ P(u,x)=S_X(\phi,\mathbf{y}(x))$.

Dually, $v\in X$ is an *infimum* of ϕ, denoted by $v=\inf\phi$, if

(1) $P(v,v)=t\phi$,
(2) $\forall x\in X,\ P(x,v)=S_X^{\mathrm{op}}(\mathbf{y}^*(x),\phi)$.

The preordered L-subset $((X,\lambda),P)$ is called *complete* if every $\phi\in\mathcal{P}(X,\lambda)$ has both a supremum and an infimum.

If $u=\sup\phi$, $\phi\in\mathcal{P}(X,\lambda)$, then for all $x\in X$,

$$P(u,x)=S_X(\phi,\mathbf{y}(x))=t\phi\wedge\lambda(x)\wedge\bigwedge_{y\in X}\Big((t\phi\to\phi(y))\to P(y,x)\Big).$$

This formula can be interpreted as the statement that u is smaller than or equal to x if and only if for each y in ϕ, y is smaller than or equal to x. Said differently, u is the smallest upper bound of ϕ, i.e., the supremum of ϕ.

Dually, $\inf\phi$ can be interpreted as the greatest lower bound of ϕ, i.e., the infimum of ϕ.

The following proposition is a special case of a general result about enriched categories in [15].

Proposition 2.11. *Suppose $((X,\lambda),P)$ is a preordered L-subset. The following conditions are equivalent:*

(1) *(X,P) is complete.*
(2) *Every $\phi\in\mathcal{P}(X,\lambda)$ has a supremum.*
(3) *Every $\psi\in\mathcal{P}^*(X,\lambda)$ has an infimum.*
(4) *Every $\phi\in\mathcal{P}(X,P)$ has a supremum.*
(5) *Every $\psi\in\mathcal{P}^*(X,P)$ has an infimum.*
(6) *The Yoneda embedding $\mathbf{y}$ has a left adjoint (given by $\sup:\mathcal{P}(X,P)\longrightarrow(X,P)$).*
(7) *The co-Yoneda embedding $\mathbf{y}^*$ has a right adjoint (given by $\inf:\mathcal{P}^*(X,P)\longrightarrow(X,P)$).*
(8) *The map $(X,P)\xrightarrow{\mathbf{y}}\mathcal{P}(X,P)\hookrightarrow\mathcal{P}(X,\lambda)$ has a left adjoint (given by $\sup:\mathcal{P}(X,\lambda)\longrightarrow(X,P)$).*
(9) *The map $(X,P)\xrightarrow{\mathbf{y}^*}\mathcal{P}^*(X,P)\hookrightarrow\mathcal{P}^*(X,\lambda)$ has a right adjoint (given by $\inf:\mathcal{P}^*(X,\lambda)\longrightarrow(X,P)$).*

Example 2.12. [15] For each preordered L-subset (X,P), both $\mathcal{P}(X,P)$ and $\mathcal{P}^*(X,P)$ are complete. In particular, both the fuzzy powerset and the dual fuzzy powerset of an L-subset (X,λ) are complete.

Definition 2.13. A closure operator on a preordered L-subset (X,P) is an order-preserving map $C:(X,P)\longrightarrow(X,P)$ such that $C\circ C=C$ and $P(x,x)\leq P(x,C(x))$ for all $x\in X$.

Example 2.14. Let $F\dashv G:(X,P)\rightharpoonup(Y,Q)$ be an adjunction between preordered L-subsets. Then $G\circ F:(X,P)\longrightarrow(X,P)$ is a closure operator.

For an L-subset (X,λ) and a subset $A\subseteq X$, we identify $\phi\in\mathcal{P}(A,\lambda)$ with $\psi\in\mathcal{P}(X,\lambda)$ for convenience, where

$$\psi(x)=\begin{cases}\phi(x), & x\in A,\\ 0, & x\notin A.\end{cases}$$

Proposition 2.15. *Let (X,P) be a complete preordered L-subset and $C:(X,P)\longrightarrow(X,P)$ a closure operator, then $(C(X),P)$ is also a complete preordered L-subset.*

Proof. Let $\lambda(x)=P(x,x)$ for all $x\in X$. For each $\phi\in\mathcal{P}^*(C(X),\lambda)$, it can be verified that the infimum of ϕ in (X,P) belongs to $C(X)$ and is the infimum of ϕ in $(C(X),P)$, i.e.,

$$\inf_{(C(X),P)}\phi=\inf_{(X,P)}\phi.$$

Thus, the conclusion follows from Proposition 2.11. □

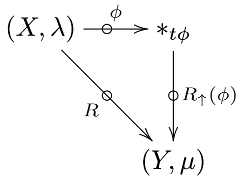
3. Formal concept analysis on L-subsets

A *fuzzy context* is a triple $((X,\lambda),(Y,\mu),R)$, where (X,λ) and (Y,μ) are L-subsets, and $R:(X,\lambda)\relbar\!\!\circ\!\!\rightarrow(Y,\mu)$ is a fuzzy relation.

Given a fuzzy context $((X,\lambda),(Y,\mu),R)$, define

$$R_{\uparrow}:\mathcal{P}(X,\lambda)\longrightarrow\mathcal{P}^*(Y,\mu)$$

as follows: for each $\phi:(X,\lambda)\relbar\!\!\circ\!\!\rightarrow *_{t\phi}$ in $\mathcal{P}(X,\lambda)$, $R_{\uparrow}(\phi)$ is the greatest fuzzy relation $*_{t\phi}\relbar\!\!\circ\!\!\rightarrow(Y,\mu)$ such that $R_{\uparrow}(\phi)\circ\phi\leq R$.



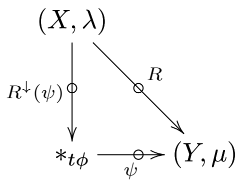
It is easy to see that for each $y\in Y$,

$$R_{\uparrow}(\phi)(y)=S_X(\phi,R(-,y)).$$

Dually, define

$$R^{\downarrow}:\mathcal{P}^*(Y,\mu)\longrightarrow\mathcal{P}(X,\lambda)$$

as follows: for each $\psi:*_{t\psi}\relbar\!\!\circ\!\!\rightarrow(Y,\lambda)$ in $\mathcal{P}^*(Y,\mu)$, $R^{\downarrow}(\psi)$ is the greatest fuzzy relation $(X,\lambda)\relbar\!\!\circ\!\!\rightarrow *_{t\psi}$ such that $\psi\circ R^{\downarrow}(\psi)\leq R$.



It can be verified that

$$R^{\downarrow}(\psi)(x)=S_Y^{\rm op}(R(x,-),\psi)$$

for all $x\in X$.

Proposition 3.1. *Each fuzzy relation* $R:(X,\lambda)\relbar\!\!\circ\!\!\rightarrow(Y,\mu)$ *gives rise to an adjunction* $R_{\uparrow}\dashv R^{\downarrow}:\mathcal{P}(X,\lambda)\rightharpoonup\mathcal{P}^*(Y,\mu)$.

For a fuzzy context $((X,\lambda),(Y,\mu),R)$, let $\mathfrak{B}(R)$ denote the set of pairs

$$(\phi,\psi)\in\mathcal{P}(X,\lambda)\times\mathcal{P}^*(Y,\mu)$$

such that

$$\psi=R_{\uparrow}(\phi)\text{ and }\phi=R^{\downarrow}(\psi).$$

It follows from the definition that $t\phi=t\psi$ for all $(\phi,\psi)\in\mathfrak{B}(R)$.

Let $\Gamma:\mathfrak{B}(R)\longrightarrow L$ be given by $\Gamma(\phi,\psi)=t\phi$. Then the L-subset $(\mathfrak{B}(R),\Gamma)$ is called the fuzzy set of formal concepts of the fuzzy context $((X,\lambda),(Y,\mu),R)$. Elements (ϕ,ψ) in $\mathfrak{B}(R)$ are called potential formal concepts of $((X,\lambda),(Y,\mu),R)$, and $\Gamma(\phi,\psi)$ is interpreted as the degree that (ϕ,ψ) is a formal concept.

For any potential formal concepts (ϕ_1,ψ_1) and (ϕ_2,ψ_2) of $((X,\lambda),(Y,\mu),R)$, let

$$\mathcal{B}((\phi_1,\psi_1),(\phi_2,\psi_2))=S_X(\phi_1,\phi_2)=S_Y^{\rm op}(\psi_1,\psi_2).$$

Then $\mathcal{B}$ is a separated preorder on the L-subset $(\mathfrak{B}(R),\Gamma)$ and $(\mathfrak{B}(R),\mathcal{B})$ is called the preordered L-subset of formal concepts in $((X,\lambda),(Y,\mu),R)$. The pair $(\mathfrak{B}(R),\mathcal{B})$ will be abbreviated to $\mathfrak{B}(R)$ if there would be no confusion.

The projection

$$\begin{array}{rccc}\pi_1: & \mathfrak{B}(R) & \longrightarrow & \mathcal{P}(X,\lambda),\\ & (\phi,\psi) & \mapsto & \phi\end{array}$$

is fully faithful. Since the image of π_1 is exactly the set of fixed points of the closure operator

$$R^{\downarrow} \circ R_{\uparrow} : \mathcal{P}(X,\lambda) \longrightarrow \mathcal{P}(X,\lambda),$$

it follows from Example 2.12 and Proposition 2.15 that $\mathfrak{B}(R)$ is isomorphic to the complete preordered L-subset $R^{\downarrow} \circ R_{\uparrow}(\mathcal{P}(X,\lambda))$. Hence $\mathfrak{B}(R)$ is a complete and separated preordered L-subset.

Proposition 3.2. *Let (X,λ) be an L-subset and $C : \mathcal{P}(X,\lambda) \longrightarrow \mathcal{P}(X,\lambda)$ a closure operator, then there is a fuzzy context $((X,\lambda),(Y,\mu),R)$ such that $C = R^{\downarrow} \circ R_{\uparrow}$.*

Proof. Let $Y = C(\mathcal{P}(X,\lambda))$ and $\mu(\phi) = S_X(\phi,\phi)$ for all $\phi \in Y$, then (Y,μ) is an L-subset. Let $R(x,\phi) = \phi(x)$ for $x \in X$ and $\phi \in Y$, then $R : (X,\lambda) \looparrowright (Y,\mu)$ is a fuzzy relation. Then $((X,\lambda),(Y,\mu),R)$ is the desired fuzzy context. □

The following theorem extends the fundamental theorem in formal concept analysis to the fuzzy setting.

Theorem 3.3. *For any fuzzy context $((X,\lambda),(Y,\mu),R)$, $\mathfrak{B}(R)$ is a complete and separated preordered L-subset. Conversely, each separated and complete preordered L-subset (X,P) is isomorphic to $\mathfrak{B}(R)$ for some fuzzy context $((X,\lambda),(Y,\mu),R)$.*

Proof. Let $\lambda(x) = P(x,x)$ for all $x \in X$. Since

$$\sup \dashv \mathbf{y} : \mathcal{P}(X,\lambda) \rightharpoonup (X,P),$$

it follows that $\mathbf{y} \circ \sup$ is a closure operator on $\mathcal{P}(X,\lambda)$. So, by virtue of Proposition 3.2, it suffices to show that (X,P) is isomorphic to $\mathbf{y} \circ \sup(\mathcal{P}(X,\lambda))$.

For each $x \in X$, since $\mathbf{y}(x) = \mathbf{y} \circ \sup \circ \mathbf{y}(x)$, then $\mathbf{y}(x) \in \mathbf{y} \circ \sup(\mathcal{P}(X,\lambda))$. Let

$$f : (X,P) \longrightarrow \mathbf{y} \circ \sup(\mathcal{P}(X,\lambda))$$

be the map that sends x to $\mathbf{y}(x)$. Then it follows from the Yoneda lemma that f is fully faithful, thus injective since (X,P) is separated. It is clear that f is surjective, hence an isomorphism. □

Let $f : ((X,\lambda),P) \longrightarrow ((Y,\mu),Q)$ be an order-preserving map. For each $\phi : (X,\lambda) \looparrowright *_{t\phi}$ in $\mathcal{P}(X,P)$, the image $f^{\rightarrow}(\phi)$ of ϕ is defined to be the fuzzy relation

$$\phi \circ f^{\natural} : (Y,\mu) \looparrowright (X,\lambda) \looparrowright *_{t\phi},$$

where $f^{\natural} : (Y,\mu) \looparrowright (X,\lambda)$ (called the cograph of f) is given by $f^{\natural}(y,x) = Q(y,f(x))$. For each $\psi : *_{t\psi} \looparrowright (X,\lambda)$ in $\mathcal{P}^*(X,P)$, the image $f^{\rightarrow}(\psi)$ of ψ is the fuzzy relation

$$f_{\natural} \circ \psi : *_{t\psi} \looparrowright (X,\lambda) \looparrowright (Y,\mu),$$

where $f_{\natural} : (X,\lambda) \looparrowright (Y,\mu)$ (called the graph of f) is given by $f_{\natural}(x,y) = Q(f(x),y)$.

Definition 3.4. Let $f : (X,P) \longrightarrow (Y,Q)$ be an order-preserving map between preordered L-subsets.

(1) f is said to be sup-*dense* if for each $y \in Y$, $y = \sup f^{\rightarrow}(\phi)$ for some $\phi \in \mathcal{P}(X,P)$.
(2) f is said to be inf-*dense* if for each $y \in Y$, $y = \inf f^{\rightarrow}(\psi)$ some $\psi \in \mathcal{P}^*(X,P)$.

Example 3.5. For each preordered L-subset (X,P), the Yoneda embedding

$$\mathbf{y} : (X,P) \longrightarrow \mathcal{P}(X,P)$$

is sup-dense and the co-Yoneda embedding

$$\mathbf{y}^* : (X,P) \longrightarrow \mathcal{P}^*(X,P)$$

is inf-dense.

Theorem 3.6. *A separated and complete preordered L-subset (W,P) is isomorphic to the complete preordered L-subset $\mathfrak{B}(R)$ of some fuzzy context $((X,\lambda),(Y,\mu),R)$ if and only if there exists a* sup-*dense map $f : (X,\lambda) \longrightarrow (W,P)$ and an* inf-*dense map $g : (Y,\mu) \longrightarrow (W,P)$ such that $R(x,y) = P(f(x),g(y))$ for all $x \in X$, $y \in Y$.*

Proof. **Necessity.** It suffices to prove the case that $(W, P) = \mathfrak{B}(R)$. Define maps $f : (X, \lambda) \longrightarrow (W, P)$ and $g : (Y, \mu) \longrightarrow (W, P)$ by

$$f(x) = (R^{\downarrow} \circ R_{\uparrow} \circ \mathbf{y}_{(X,\lambda)}(x), R_{\uparrow} \circ \mathbf{y}_{(X,\lambda)}(x)),$$
$$g(y) = (R^{\downarrow} \circ \mathbf{y}^{*}_{(Y,\mu)}(y), R_{\uparrow} \circ R^{\downarrow} \circ \mathbf{y}^{*}_{(Y,\mu)}(y)),$$

then f, g satisfy the conditions.

Sufficiency. The map

$$\begin{array}{ccc} (W, P) & \longrightarrow & \mathfrak{B}(R) \\ w & \mapsto & \big(P(f(-), w), P(w, g(-))\big) \end{array}$$

is the desired isomorphism. □

4. Conclusion

The machinery of Formal Concept Analysis is extended to the realm of fuzzy relations between fuzzy sets in the case that the truth-table $(L, \&)$ is a divisible, commutative complete residuated lattice. It is shown that the fundamental theorem of FCA is also valid in this setting.

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