多值状态属性系统与闭包空间

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摘要: 本文把状态属性系统的概念推广到了多值情形, 探讨了状态属性系统与闭包空间之间的关系. 证明了对一个有单位元的 quantale L, 在 L 中取值的状态属性系统的范畴与在 L 中取值的闭包空间范畴等价.

关键词: quantale; L-完备格; L-状态属性系统; L-闭包空间

中图分类号：O159 文献标识码：A

Many-valued state property systems and closure spaces

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Abstract: In this paper the definition of state property system is extended to the many-valued setting. The relationship between state property systems and closure spaces is discussed. It is shown that, for a unital quantale L, the category of state property systems valued in L is equivalent to the category of closure spaces valued in L.

Key Words: quantale, complete L-lattice, L-state property system, L-closure space

1 Introduction

A *state property system* [1–3] is a triple (X, Y, R), where X is a preordered set, Y is a complete lattice, and and $R \subseteq X \times Y$ is a relation from X to Y. In a state property system (X, Y, R), X is considered to be the set of *states*, Y the set of *tests*, and $(x, y) \in R$ means that y is true if the state is x. A *morphism* [1] $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between state property systems is a pair of functions $f : X \longrightarrow A$ and $g : B \longrightarrow Y$ such that $(x, g(b)) \in R$ if and only if $(f(x), b) \in S$ for all $x \in X$ and $b \in B$.

It is shown in Reference [1] that the category of state property systems is equivalent to the category of closure spaces and continuous functions. In this note we will extend this conclusion to the quantale-valued setting. Precisely, for a unital quantale $(L, *)$, the notion of L-state property systems is introduced and it is proved that the category of L-state property systems is equivalent to the category of L-closure spaces.

基金项目: 国家自然科学基金(11071174)
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2 Basic notions

A *quantale* [4–6] is a pair $(L, *)$, where L is a complete lattice, $*$ is an associative binary operation on L such that $a * (\bigvee b_i) = \bigvee(a * b_i)$ and $(\bigvee b_i) * a = \bigvee(b_i * a)$ for all $a, b_i \in L$. The top and the bottom element of L is denoted by 1 and 0 respectively. A quantale $(L, *)$ is said to be *unital* if there exists an element $I \in L$ such that $I * a = a = a * I$ for all $a \in L$.

Definition 2.1 [4] *Let $(L, *)$ be a quantale. Define $\rightarrow_l, \rightarrow_r$: $L \times L \longrightarrow L$ by*

$$b \rightarrow_l c = \bigvee\{a \in L : a * b \leq c\};$$

$$b \rightarrow_r c = \bigvee\{a \in L : b * a \leq c\}.$$

Proposition 2.2 [4] *Let $(L, *)$ be a quantale.*

(I1) $a * b \leq c \iff a \leq b \rightarrow_l c \iff b \leq a \rightarrow_r c$.

(I2) $a \rightarrow_l (\bigwedge b_t) = \bigwedge(a \rightarrow_l b_t)$; $a \rightarrow_r (\bigwedge b_t) = \bigwedge(a \rightarrow_r b_t)$.

(I3) $(\bigvee a_t) \rightarrow_l b = \bigwedge(a_t \rightarrow_l b)$; $(\bigvee a_t) \rightarrow_r b = \bigwedge(a_t \rightarrow_r b)$.

(I4) $(a \rightarrow_r b) * (b \rightarrow_r c) \leq a \rightarrow_r c$; $(b \rightarrow_l c) * (a \rightarrow_l b) \leq a \rightarrow_l c$.

(I5) $a \rightarrow_l (b \rightarrow_l c) = (a * b) \rightarrow_l c$; $a \rightarrow_r (b \rightarrow_r c) = (b * a) \rightarrow_r c$.

(I6) $a \rightarrow_r (b \rightarrow_l c) = b \rightarrow_l (a \rightarrow_r c)$.

For any set X, the set L^X of maps $X \longrightarrow L$ with the pointwise order is also a complete lattice. Furthermore, L^X is a quantale with the binary operation $*$ and $\rightarrow_l, \rightarrow_r$ defined pointwisely. Elements of L^X are called L-subsets of X. For sets X and Y, an L-relation from X to Y is a function $R : X \times Y \longrightarrow L$.

Given a subset Y of X and an L-subset $\lambda : Y \longrightarrow L$ of Y, we identify λ with the L-subset μ of X given by $\mu(x) = \lambda(x)$ if $x \in Y$ and $\mu(x) = 0$ otherwise.

Definition 2.3 *An L-preorder on a set X is an L-relation $P : X \times X \longrightarrow L$ such that*

(1) *$I \leq P(x, x)$ for each $x \in X$ (reflexivity);*

(2) *$P(x, y) * P(y, z) \leq P(x, z)$ for all $x, y, z \in X$ (transitivity).*

Further more, if P satisfies

(3) *if $P(x, y) \geq I$ and $P(y, x) \geq I$ then $x = y$ (anti-symmetry),*

then P is called an L-order.

In an L-preordered set (X, P), the value $P(x, y)$ can be interpreted as the *degree* to which x is less than or equal to y. We abbreviate the pair (X, P) to X and write $X(x, y)$ instead of $P(x, y)$ if there is no confusion.

Given an L-preordered set X, define a binary relation $\leq$ on X by $x \leq y \iff X(x, y) \geq I$. Then $\leq$ is reflexive and transitive, hence $\leq$ is a preorder on X [7]. The pair $(X, \leq)$ is called the underlying ordered set of X, and will be denoted by X_0. It is easy to see that X_0 is a partially ordered set if X is an L-ordered set.

A function $f : A \longrightarrow B$ between L-preordered sets is *L-order preserving* if $A(a, b) \leq B(f(a), f(b))$ for all $a, b \in A$. $f : A \longrightarrow B$ is *L-isometric* if $A(a, b) = B(f(a), f(b))$ for all $a, b \in A$.

Example 2.4 (1) *Let $L(a, b) = a \rightarrow_r b$, then L is an L-ordered set. Similarly, let $L_c^{\text{op}}(a, b) = b \rightarrow_l a$, then L_c^{op} is an L-ordered set.*

(2) *(L-powerset) Let X be a set. For all L-subsets $\mu, \lambda \in L^X$, let*

$$L^X(\mu, \lambda) = \bigwedge_{x \in X} \mu(x) \rightarrow_r \lambda(x)$$

and

$$(L_c^X)^{\text{op}}(\mu, \lambda) = \bigwedge_{x \in X} \lambda(x) \rightarrow_l \mu(x),$$

then L^X and $(L_c^X)^{\text{op}}$ are L-ordered sets.

Definition 2.5 *Let A be an L-ordered set and $\mu \in L^A$. If there is some $a \in A$ satisfying*

$$A(a, y) = \bigwedge_{x \in A} (\mu(x) \to_r A(x, y))$$

for all $y \in A$, we say that a is a supremum of μ, and denote it by $\sup \mu$. *Dually, if there is some $b \in A$ satisfying*

$$A(y, b) = \bigwedge_{x \in A} (\mu(x) \to_l A(y, x))$$

for all $y \in A$, then b is called an infimum of μ, and will be denoted by $\inf \mu$.

Definition 2.6 *An L-ordered set A is a complete L-lattice if every L-subset μ of A has both a supremum and an infimum.*

Definition 2.7 *[8] A pair of L-order preserving functions $f : A \longrightarrow B$ and $g : B \longrightarrow A$ is called an L-adjunction (denoted by $f \vdash g : A \rightharpoonup B$) if $B(f(x), y) = A(x, g(y))$ for all $x \in A, y \in B$. In this case, f is called a left adjoint of g and g a right adjoint of f.*

Example 2.8 *For any sets X, Y and function $f : X \longrightarrow Y$, $f^{\to} \vdash f^{\leftarrow} : L^X \rightharpoonup L^Y$, where $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ are given by $f^{\to}(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ and $f^{\leftarrow}(\lambda)(x) = \lambda(f(x))$.*

Proposition 2.9 [9,10] *Let A, B be complete L-lattices and $f : A \longrightarrow B$ a function.*

(1) *$f : A \longrightarrow B$ is a left adjoint if and only if f preserves supremum in the sense that $f(\sup_A \mu) = \sup_B f^{\to}(\mu)$ for all $\mu \in L^A$.*

(2) *$f : A \longrightarrow B$ is a right adjoint if and only if f preserves infimum in the sense that $f(\inf_A \mu) = \inf_B f^{\to}(\mu)$ for all $\mu \in L^A$.*

Definition 2.10 *An L-state property system is a triple (X, Y, R), where X is an L-preordered set, Y is a complete L-lattice, and R is an L-relation from X to Y, such that*

(i) $R(x, \top_Y) \geq I$ for $x \in X$, where $\top_Y$ is the top element in Y_0;

(ii) $R(x, \bot_Y) = 0$ for $x \in X$, where $\bot_Y$ is the bottom element in Y_0;

(iii) For all $x \in X$ and $\lambda \in L^Y$,

$$R(x, \inf \lambda) = \bigwedge_{y \in Y} \lambda(y) \to_l R(x, y),$$

(iv) For all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$X(x_1, x_2) = \bigwedge_{y \in Y} R(x_2, y) \to_l R(x_1, y),$$

$$Y(y_1, y_2) = \bigwedge_{x \in X} R(x, y_1) \to_r R(x, y_2).$$

Definition 2.11 *A morphism $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between L-state property systems is a pair of functions $f : X \longrightarrow A$ and $g : B \longrightarrow Y$ such that $R(x, g(b)) = S(f(x), b)$ for all $x \in X, b \in B$.*

The category of L-state property systems and morphisms is denoted by L-**Sp**, it is an extension of the category **Sp** of state property systems [1] to the many-valued setting.

Definition 2.12 *Let X be a set. An L-order preserving function $c : L^X \longrightarrow L^X$ is called an L-closure operator if*

(C1) *for all $\mu \in L^X$, $L^X(\mu, c(\mu)) \geq I$;*

(C2) *$c \circ c = c$.*

Example 2.13 *Let X, Y be sets and $f \vdash g : L^X \rightharpoonup L^Y$ be an L-adjunction. Then $g \circ f : L^X \longrightarrow L^X$ is an L-closure operator.*

Definition 2.14 *An L-closure space is a pair (X, c), where X is a set and $c : L^X \longrightarrow L^X$ is an L-closure operator. A continuous function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces is a function $f : X \longrightarrow Y$ such that $f^{\to} \circ c(\mu) \leq d \circ f^{\to}(\mu)$ for all $\mu \in L^X$.*

The category of L-closure spaces and continuous functions is denoted by L-**Cls**.

3 The equivalence of L-Sp and L-Cls

Let (X, Y, R) be an L-state property system, define a pair of operators

$$R_\uparrow : L^X \longrightarrow L^Y, \quad R^\downarrow : L^Y \longrightarrow L^X$$

by

$$R_\uparrow(\mu)(y) = \bigwedge_{x\in X} (\mu(x) \rightarrow_r R(x,y));$$

$$R^\downarrow(\lambda)(x) = \bigwedge_{y\in Y} (\lambda(y) \rightarrow_l R(x,y)).$$

Then it is easy to see that

$$(L_c^Y)^{\mathrm{op}}(R_\uparrow(\mu), \lambda) = L^X(\mu, R^\downarrow(\lambda))$$

for all $\mu \in L^X$ and $\lambda \in L^Y$. Hence

$$R_\uparrow \vdash R^\downarrow : L^X \rightharpoonup (L_c^Y)^{\mathrm{op}}$$

is an L-adjunction*. Therefore, $R^\downarrow \circ R_\uparrow : L^X \longrightarrow L^X$ is an L-closure operator.

Proposition 3.1 [14] *Let $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ be a morphism between L-state property systems. Then $f : (X, R^\downarrow \circ R_\uparrow) \longrightarrow (A, S^\downarrow \circ S_\uparrow)$ is continuous.*

Proposition 3.1 gives rise to a functor $U : L$-**Sp** $\longrightarrow L$-**Cls** that sends a morphism

$$(f, g) : (X, Y, R) \longrightarrow (A, B, S)$$

to the continuous function

$$f : (X, R^\downarrow \circ R_\uparrow) \longrightarrow (A, S^\downarrow \circ S_\uparrow).$$

Given a continuous function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces, let $c(L^X)$ and $d(L^Y)$ denote respectively the sets of all closed L-subsets of X and Y. Let

$$R'(x, \lambda) = \lambda(x), \quad S'(y, \mu) = \mu(y)$$

for all $x \in X$, $\lambda \in c(L^X)$, and $y \in Y$, $\mu \in d(L^Y)$. For all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, let

$$\begin{aligned} X(x_1, x_2) &= \bigwedge_{\lambda\in c(L^X)} R'(x_2, \lambda) \rightarrow_l R'(x_1, \lambda) \\ &= \bigwedge_{\lambda\in c(L^X)} \lambda(x_2) \rightarrow_l \lambda(x_1), \end{aligned}$$

$$\begin{aligned} Y(y_1, y_2) &= \bigwedge_{\mu\in d(L^Y)} S'(y_2, \mu) \rightarrow_l S'(y_1, \mu) \\ &= \bigwedge_{\mu\in d(L^Y)} \mu(y_2) \rightarrow_l \mu(y_1). \end{aligned}$$

Then $(X, c(L^X), R')$ and $(Y, d(L^Y), S')$ are L-state property systems†.

Consider the function $f^* : d(L^Y) \longrightarrow c(L^X)$ that sends each closed L-subset λ of Y to $f^*(\lambda) = f^\leftarrow(\lambda)$. Then we have the following

Proposition 3.2 *$(f, f^*) : (X, c(L^X), R') \longrightarrow (Y, d(L^Y), S')$ is a morphism between L-state property systems.*

Proof For all $x \in X$ and $\lambda \in d(L^Y)$, we have that

$$\begin{aligned} R'(x, f^*(\lambda)) &= f^*(\lambda)(x) \\ &= f^\leftarrow(\lambda)(x) \\ &= \lambda(f(x)) \\ &= S'(f(x), \lambda), \end{aligned}$$

the conclusion thus follows.

From the above proposition we obtain a functor

$$F : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Sp}$$

that sends a continuous function $f : (X, c) \longrightarrow (Y, d)$ to the morphism

$$(f, f^*) : (X, c(L^X), R') \longrightarrow (Y, d(L^Y), S').$$

*When L is a complete residuated lattice, the pair $(R_\uparrow, R^\downarrow)$ is introduced by Bělohlávek first and is called a fuzzy Galois connection [11–13].

†The L-preorders on X and Y are the specialization order of the L-closure spaces (X, c) and (Y, d).

Proposition 3.3 [14] *For each L-closure space (X, c), $U \circ F(X, c) = (X, c)$.*

Theorem 3.4 *The functors $F : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Sp}$ and $U : L\text{-}\mathbf{Sp} \longrightarrow L\text{-}\mathbf{Cls}$ establish an equivalence of categories.*

Proof The proof is divided into two steps.

Step 1. $F \circ U \cong \mathbf{id}_{L\text{-}\mathbf{Sp}}$.

For an L-state property system (X, Y, R), $F \circ U(X, Y, R) = (X, R^{\downarrow} \circ R_{\uparrow}(L^X), R')$ by definition, where R' is defined as in Proposition 3.2. For a morphism $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between L-state property systems, $F \circ U(f, g) = (f, f^*)$ is a morphism from $(X, R^{\downarrow} \circ R_{\uparrow}(L^X), R')$ to $(A, S^{\downarrow} \circ S_{\uparrow}(L^A), S')$.

For each L-state property system (X, Y, R), define

$$\eta_{(X,Y,R)} = (\mathrm{id}_X, h_Y) : F \circ U(X, Y, R) \longrightarrow (X, Y, R)$$

by $h_Y(y) = R^{\downarrow}(I_y)$, where $I_y \in L^Y$ is given by $I_y(y) = I$ and $I_y(y') = 0$ whenever $y' \neq y$. We claim that η is a natural isomorphism from $F \circ U$ to the identity functor.

Firstly, $h_Y(y) \in R^{\downarrow} \circ R_{\uparrow}(L^X)$ is clear, and for each $x \in X$ and $y \in Y$,

$$R(x, y) = R^{\downarrow}(I_y)(x) = R'(x, R^{\downarrow}(I_y)) = R'(x, h_Y(y)).$$

Hence $\eta_{(X,Y,R)}$ is a morphism.

Secondly, we prove that $\eta_{(X,Y,R)}$ is an isomorphism. For this purpose, it suffices to show that $h_Y : Y \longrightarrow R^{\downarrow} \circ R_{\uparrow}(L^X)$ is bijective. Since

$$\begin{aligned} Y(y_1, y_2) &= \bigwedge_{x \in X} R(x, y_1) \to_r R(x, y_2) \\ &= L^X(h_Y(y_1), h_Y(y_2)) \end{aligned}$$

for all $y_1, y_2 \in Y$, it follows that h_Y is L-isometric, and then injective. Given $\mu \in L^X$, let $y = \inf R_{\uparrow}(\mu)$. Then for all $x \in X$,

$$\begin{aligned} h_Y(y)(x) &= R(x, y) \\ &= \bigwedge_{y \in Y} R_{\uparrow}(\mu)(y) \to_l R(x, y) \\ &= R^{\downarrow} \circ R_{\uparrow}(\mu)(x), \end{aligned}$$

i.e. $h_Y(y) = R^{\downarrow} \circ R_{\uparrow}(\mu)$, hence h_Y is surjective.

Thirdly, the naturality of η follows from the commutativity of the following diagram for any morphism $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$:

$$\begin{array}{ccc} F \circ U(X, Y, R) & \xrightarrow{(\mathrm{id}_X, h_Y)} & (X, Y, R) \\ \Big\downarrow {\scriptstyle (f, f^*)} & & \Big\downarrow {\scriptstyle (f, g)} \\ F \circ U(A, B, S) & \xrightarrow[(\mathrm{id}_A, h_B)]{} & (A, B, S) \end{array}$$

In fact, $f \circ \mathrm{id}_X = \mathrm{id}_A \circ f$ is clear, and for all $x \in X$ and $b \in B$,

$$\begin{aligned} h_Y(g(b))(x) &= R(x, g(b)) \\ &= S(f(x), b) \\ &= f^* \circ h_B(b)(x), \end{aligned}$$

thus the conclusion follows.

Step 2. $U \circ F \cong \mathbf{id}_{L\text{-}\mathbf{Cls}}$.

For each L-closure space (X, c), $\mathrm{id}_{(X,c)} : (X, c) \longrightarrow U \circ F(X, c)$ is clearly continuous and $\{\mathrm{id}_{(X,c)}\}$ is a natural isomorphism from $\mathbf{id}_{L\text{-}\mathbf{Cls}}$ to $U \circ F$ by Proposition 3.3.

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