THE CONCEPT LATTICE FUNCTORS

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Abstract. This paper is concerned with the relationship between contexts, closure spaces, and complete lattices. It is shown that, for a unital quantale L, both formal concept lattices and property oriented concept lattices are functorial from the category L-**Ctx** of L-contexts and infomorphisms to the category L-**Sup** of complete L-lattices and suprema-preserving maps. Moreover, the formal concept lattice functor can be written as the composition of a right adjoint functor from L-**Ctx** to the category L-**Cls** of L-closure spaces and continuous functions and a left adjoint functor from L-**Cls** to L-**Sup**.

1. Introduction

A *formal context* is a triple (X, Y, R), where X, Y are sets and $R \subseteq X \times Y$ is a relation from X to Y. In a formal context (X, Y, R), X is considered to be the set of *objects*, Y the set of *properties*, and $(x, y) \in R$ means that the object x has the property y. Formal contexts provide a common framework for formal concept analysis (FCA) [7, 10] and rough set theory (RST) [11, 26]. Given a context (X, Y, R), there exists a contravariant Galois connection $(R_\uparrow, R^\downarrow)$ and a covariant Galois connection $(R_\exists, R^\forall)$ between the powersets of X and Y. These two Galois connections play fundamental roles in formal concept analysis and rough set theory respectively.

A *formal concept* [10] of the context (X, Y, R) is a pair $(U, V) \in 2^X \times 2^Y$ satisfying $U = R^\downarrow(V)$ and $V = R_\uparrow(U)$; a *property oriented concept* [11, 26, 27] of the context (X, Y, R) is a pair $(U, V) \in 2^X \times 2^Y$ satisfying $U = R^\forall(V)$ and $V = R_\exists(U)$. The set of all the formal concepts of the context (X, Y, R) is denoted by $\mathfrak{B}(X, Y, R)$, and the set of all the property oriented concepts by $\mathfrak{P}(X, Y, R)$. Both $\mathfrak{B}(X, Y, R)$ and $\mathfrak{P}(X, Y, R)$ are complete lattices.

This paper is concerned with the functorial properties of $\mathfrak{B}$ and $\mathfrak{P}$ (see [19] for category theory). To this end, we must determine the morphisms between contexts and that between complete lattices. There are different approaches to morphisms between contexts, see e.g. [8, 9, 10, 15, 16, 20, 25, 28].[1] In particular, Mori [20] has shown that the construction of formal concept lattices induces an equivalence between the category of contexts and Chu correspondences and that of complete lattices and suprema-preserving maps.

In this paper, we consider infomorphisms between formal contexts. It is shown that both the construction of formal concept lattices and that of property oriented concept lattices are functorial from the category **Ctx** of contexts and infomorphisms to the category **Sup** of complete lattices and suprema-preserving maps. It should be noted that an infomorphism is not a Chu correspondence in the sense of Mori [20] in general, hence the functoriality discussed here does not follow from the result of Mori.

An infomorphism [9, 16] $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between contexts is a pair of functions $f : X \longrightarrow A$ and $g : B \longrightarrow Y$ such that $(x, g(b)) \in R$ if and only if $(f(x), b) \in S$ for all $x \in X$ and $b \in B$. Contexts and infomorphisms constitute a category **Ctx**.

Key words and phrases. Formal concept analysis, Rough set theory, Concept lattice, Unital quantale, Complete L-lattice, L-closure space.

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[1] In rough set theory, there exist different approaches to morphisms between formal contexts of the form (X, X, R) with a subset $A \subseteq X$, where R is an equivalence relation on X, see Banerjee and Chakraborty [1], Banerjee and Yao [2] for instance. We are grateful to one of the reviewers for bringing [1, 2] to our attention.

Though it is possible to show that $\mathfrak{B}, \mathfrak{P}$ are both functorial from **Ctx** to **Sup** in a direct way, we will establish the functoriality of $\mathfrak{B}, \mathfrak{P}$ by help of closure spaces. The benefit of doing so is that we will obtain decompositions of the functors $\mathfrak{B}$ and $\mathfrak{P}$, these decompositions are helpful for further investigation on these functors and the interrelationship between contexts, closure spaces, and complete lattices.

Let X be a set, a closure operator is an order-preserving map $c : 2^X \longrightarrow 2^X$ on the powerset of X such that $A \subseteq c(A)$ for all $A \subseteq X$ and $c \circ c = c$. The pair (X, c) is called a closure space, a subset $A \subseteq X$ is closed if $A = c(A)$. A map $f : (X, c) \longrightarrow (Y, d)$ between closure spaces is continuous if $f(c(A)) \subseteq d(f(A))$ for each subset A of X. The category of closure spaces and continuous maps is denoted by **Cls**.

Given a context (X, Y, R), both $R^{\downarrow} \circ R_{\uparrow}$ and $R^{\forall} \circ R_{\exists}$ are closure operators on X. The correspondence $(X, Y, R) \mapsto (X, R^{\downarrow} \circ R_{\uparrow})$ defines a functor $U : \mathbf{Ctx} \longrightarrow \mathbf{Cls}$ which has a left adjoint.

Given a closure space (X, c), the set $c(2^X)$ of all the closed subsets of (X, c) is a complete lattice. The correspondence $(X, c) \mapsto c(2^X)$ defines a functor $T : \mathbf{Cls} \longrightarrow \mathbf{Sup}$ which has a right adjoint.

For each context (X, Y, R), $T \circ U(X, Y, R)$ is isomorphic to the formal concept lattice $\mathfrak{B}(X, Y, R)$. Hence $\mathfrak{B}$ is functorial from **Ctx** to **Sup**, and it is the composition of a right adjoint functor $U : \mathbf{Ctx} \longrightarrow \mathbf{Cls}$ and a left adjoint functor $T : \mathbf{Cls} \longrightarrow \mathbf{Sup}$.

The property oriented concept lattice functor can also be written as a composition of a functor $V : \mathbf{Ctx} \longrightarrow \mathbf{Cls}$ and the functor $T : \mathbf{Cls} \longrightarrow \mathbf{Sup}$.

All the conclusions stated above will be proved in a much more general setting in this paper. The theories of formal concept lattices and property oriented concept lattices have been generalized to the fuzzy setting [6, 11, 12, 13, 17, 21, 26]. We shall prove the L-version of the conclusions stated above for a unital quantale $(L, \&)$.

The contents are arranged as follows. Section 2 recalls some basic notions of quantales. Section 3 introduces the categories considered in this paper: the category L-**Ctx** of L-contexts and infomorphisms, the category L-**Sup** of complete L-lattices and suprema-preserving maps, and the category L-**Cls** of L-closure spaces and continuous maps. An adjunction between L-**Cls** and L-**Sup** is given in Section 4. Section 5 presents an adjunction between L-**Ctx** and L-**Cls**. In Section 6, it is demonstrated that both the formal concept lattices and property oriented concept lattices of L-contexts are functorial from L-**Ctx** to L-**Sup**, and the formal concept lattice functor is the composition of a right adjoint functor and a left adjoint functor. And, if $(L, \&)$ has a dualizing element, then the property oriented concept lattice functor is the composition of the formal concept lattice functor following a functor L-**Ctx** $\longrightarrow$ L-**Ctx** and vice versa.

2. Quantales

A *quantale* [22] is a pair $(L, \&)$, where L is a complete lattice, $\&$ is an associative binary operation on L such that $a \& (\bigvee b_i) = \bigvee (a \& b_i)$ and $(\bigvee b_i) \& a = \bigvee (b_i \& a)$ for all $a, b_i \in L$. The top and the bottom element of L is denoted by 1 and 0 respectively. A quantale $(L, \&)$ is said to be *unital* if there exists an element $I \in L$ such that $I \& a = a = a \& I$ for all $a \in L$. Finally, $(L, \&)$ is commutative if $a \& b = b \& a$ for all $a, b \in L$.

Definition 2.1. [22] Let $(L, \&)$ be a quantale. Define $\swarrow, \searrow : L \times L \longrightarrow L$ by

$$c \swarrow b = \bigvee \{a \in L : a \& b \leq c\} \text{ and } b \searrow c = \bigvee \{a \in L : b \& a \leq c\}.$$

If $(L, \&)$ is commutative, then $c \swarrow b = b \searrow c$ for all $b, c \in L$ and will be denoted by $b \rightarrow c$.

Proposition 2.2. [22] *Let $(L, \&)$ be a quantale. The following properties hold for all $a, b, c, a_t, b_t \in L$:*

(1) $a \leq c \swarrow b \iff a \& b \leq c \iff b \leq a \searrow c$.
(2) $(\bigwedge b_t) \swarrow a = \bigwedge (b_t \swarrow a)$; $a \searrow (\bigwedge b_t) = \bigwedge (a \searrow b_t)$.
(3) $b \swarrow (\bigvee a_t) = \bigwedge (b \swarrow a_t)$; $(\bigvee a_t) \searrow b = \bigwedge (a_t \searrow b)$.
(4) $(a \searrow b) \& (b \searrow c) \leq a \searrow c$; $(c \swarrow b) \& (b \swarrow a) \leq c \swarrow a$.
(5) $(c \swarrow b) \swarrow a = c \swarrow (a \& b)$; $a \searrow (b \searrow c) = (b \& a) \searrow c$.

(6) $a \searrow (c \swarrow b) = (a \searrow c) \swarrow b$.
(7) $a \& (a \searrow b) \leq b$; $(b \swarrow a) \& a \leq b$.

An element d in a quantale $(L, \&)$ is *cyclic* [22] if $d \swarrow a = a \searrow d$ for all $a \in L$. In this case, we write $a \to d$ for $d \swarrow a = a \searrow d$. It is easy to check that a quantale $(L, \&)$ is commutative if and only if every element of L is cyclic. An element d in $(L, \&)$ is *dualizing* [22] if $d \swarrow (a \searrow d) = a = (d \swarrow a) \searrow d$ for all $a \in L$.

Definition 2.3. [22] A *Girard quantale* is a quantale with a cyclic dualizing element.

A Girard quantale is necessarily unital since $d \to d$ is easily verified to be a unit element, where d is a cyclic dualizing element.

Proposition 2.4. *If d is a dualizing element in a quantale $(L, \&)$, then the following properties hold for all $a, b, a_t \in L$:*

(8) $\bigvee (d \swarrow a_t) = d \swarrow (\bigwedge a_t)$; $\bigvee (a_t \searrow d) = (\bigwedge a_t) \searrow d$.
(9) $b \swarrow a = (d \swarrow b) \searrow (d \swarrow a)$; $a \searrow b = (a \searrow d) \swarrow (b \searrow d)$.
(10) $b \& a = d \swarrow \big(a \searrow (b \searrow d)\big)$; $a \& b = \big((d \swarrow b) \swarrow a\big) \searrow d$.
(11) $a \searrow b = \big((d \swarrow b) \& a\big) \searrow d$; $b \swarrow a = d \swarrow \big(a \& (b \searrow d)\big)$.
(12) $(d \swarrow a) \searrow b = a \swarrow (b \searrow d)$.

Given a quantale $(L, \&)$, the *conjugate* of $(L, \&)$ is the quantale $(L, \bullet)$ with $a \bullet b = b \& a$ for all $a, b \in L$. Throughout this paper, L always denotes a unital quantale and L_c the conjugate of L if not otherwise specified. It is clear that $L_c = L$ when L is commutative.

For any set X, the set L^X of maps $X \longrightarrow L$ with the pointwise order is also a complete lattice. Furthermore, L^X is a quantale with the binary operation $\&$ and $\swarrow, \searrow$ defined pointwisely. Elements of L^X are called L-subsets (or, fuzzy subsets) of X. For two sets X and Y, an L-relation (or, a fuzzy relation) from X to Y is a function $R : X \times Y \longrightarrow L$. Each L-relation $R : X \times Y \longrightarrow L$ has a dual L-relation R^{op} from Y to X with $R^{\mathrm{op}}(y, x) = R(x, y)$ for all $x \in X$ and $y \in Y$.

Given a subset Y of X and an L-subset $\lambda : Y \longrightarrow L$ of Y, we identify λ with the L-subset μ of X given by $\mu(x) = \lambda(x)$ if $x \in Y$ and $\mu(x) = 0$ otherwise.

3. L-contexts, complete L-lattices, and L-closure spaces

For convenience of the readers, we recall here some basic notions of L-contexts, L-orders, complete L-lattices and L-closure spaces. Most of these notions have been defined in the literature, and some of them will be generalized slightly if needed.

3.1. L-contexts.

An *L-context* (or, a fuzzy context) is a triple (X, Y, R), where R is an L-relation from X to Y. There are different approaches to morphisms between L-contexts, and the reader is referred to [8, 9, 10, 15, 16, 20, 25, 28] for a discussion in the case that $L = 2 = \{0, 1\}$. In this paper, we are concerned with only one kind of morphisms between L-contexts, namely, that of infomorphisms.

An *infomorphism* $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between L-contexts is a pair of functions $f : X \longrightarrow A$ and $g : B \longrightarrow Y$ such that $R(x, g(b)) = S(f(x), b)$ for all $x \in X, b \in B$. The category of L-contexts and infomorphisms is denoted by L-**Ctx**.

Remark 3.1. The category L-**Ctx** is clearly the $*$-autonomous completion $\mathbf{Set}_\perp$ of **Set** with $\perp = L$ (or equivalently, L-**Ctx** is the category of Chu spaces) in the sense of Barr [3]. Hence, L-**Ctx** is a complete and cocomplete category.

3.2. Complete L-lattices.

Definition 3.2. An *L-order* (or, a fuzzy order) on a set X is an L-relation $P : X \times X \longrightarrow L$ such that

(1) $I \leq P(x, x)$ for each $x \in X$ (reflexivity);
(2) $P(x, y) \& P(y, z) \leq P(x, z)$ for all $x, y, z \in X$ (transitivity);
(3) if $P(x, y) \geq I$ and $P(y, x) \geq I$ then $x = y$ (anti-symmetry).

The pair (X, P) is called an L-ordered set.

In an L-ordered set (X, P), the value $P(x, y)$ can be interpreted as the *degree* to which x is less than or equal to y. We abbreviate the pair (X, P) to X and write $X(x, y)$ instead of $P(x, y)$ if there is no confusion. It is noteworthy to point out that L-ordered sets can be treated as categories enriched over the unital quantale L [14, 23, 24].

Given an L-ordered set X, define a binary relation $\leq$ on X by $x \leq y \iff X(x, y) \geq I$. Then $\leq$ is reflexive, transitive and anti-symmetric, hence $\leq$ is a partial order on X. The pair $(X, \leq)$ is called the underlying ordered set of X, and will be denoted by X_0.

A function $f : A \longrightarrow B$ between L-ordered sets is *L-order preserving* if $A(a, b) \leq B(f(a), f(b))$ for all $a, b \in A$. $f : A \longrightarrow B$ is *L-isometric* if $A(a, b) = B(f(a), f(b))$ for all $a, b \in A$.

Let L-**Ord** denote the category of L-ordered sets and L-order preserving functions. It is clear that an L-isometry must be an injective function and surjective L-isometries are exactly the isomorphisms in L-**Ord**.

Example 3.3. Some basic examples of L-ordered sets are listed below.

(1) (The canonical L-order on L) Let $L(a, b) = a \searrow b$, then L is an L-ordered set. Similarly, L_c is an L_c-ordered set with $L_c(a, b) = b \swarrow a$.
(2) If $(L, \&) = ([0, \infty]^{\rm op}, +)$, then an L-ordered set is exactly a generalized quasi-metric space [18].
(3) If A is an L-ordered set, let $A^{\rm op}(a, b) = A(b, a)$ for all $a, b \in A$, then $A^{\rm op}$ is an L_c-ordered set, called the *dual* of A. In particular, $L^{\rm op}$ is an L_c-ordered set with $L^{\rm op}(a, b) = b \swarrow a$ and $L_c^{\rm op}$ is an L-ordered set with $L_c^{\rm op}(a, b) = a \swarrow b$.
(4) Let A be an L-ordered set and B a subset of A. Then B is an L-ordered set with the L-order inherited from A.
(5) (Discrete L-ordered set) Given a set X, let $X(x, y) = I$ if $x = y$; $X(x, y) = 0$ if $x \neq y$. Then X becomes an L-ordered set. Such L-ordered sets are called discrete L-ordered sets.
(6) (L-powerset) Let X be a set. For all L-subsets $\mu, \lambda \in L^X$, let

$$L^X(\mu, \lambda) = \bigwedge_{x \in X} \mu(x) \searrow \lambda(x)$$

and

$$L_c^X(\mu, \lambda) = \bigwedge_{x \in X} \lambda(x) \swarrow \mu(x).$$

Then L^X is an L-ordered set and L_c^X is an L_c-ordered set. Hence, $(L_c^X)^{\rm op}$ is an L-ordered set and $(L^X)^{\rm op}$ is an L_c-ordered set.

Definition 3.4. Let A be an L-ordered set. Define $\mathbf{y} : A \longrightarrow L^A$ and $\mathbf{y}' : A \longrightarrow (L_c^A)^{\rm op}$ by

$$\mathbf{y}(x) = A(-, x) \text{ and } \mathbf{y}'(x) = A(x, -).$$

$\mathbf{y}$ is called the *Yoneda embedding* and $\mathbf{y}'$ the *co-Yoneda embedding.*

Lemma 3.5. (Yoneda lemma, [14]) *Let A be an L-ordered set.*

(1) *Let $\mu : A \longrightarrow L_c^{\rm op}$ be L-order preserving, then for all $x \in X$, $L^A(\mathbf{y}(x), \mu) = \mu(x)$.*
(2) *Let $\lambda : A \longrightarrow L$ be L-order preserving, then for all $x \in X$, $L_c^A(\mathbf{y}'(x), \lambda) = \lambda(x)$.*
(3) *Both $\mathbf{y} : A \longrightarrow L^A$ and $\mathbf{y}' : A \longrightarrow (L_c^A)^{\rm op}$ are L-isometries.*

Definition 3.6. Let A be an L-ordered set and $\mu \in L^A$. The L-subset $\mathbf{ub}(\mu)$ of *upper bounds of* μ is given by

$$\mathbf{ub}(\mu)(y) = \bigwedge_{x \in A} \big(\mu(x) \searrow A(x, y)\big). \tag{1}$$

Dually, the L-subset $\mathbf{lb}(\mu)$ of *lower bounds of* μ is given by

$$\mathbf{lb}(\mu)(y) = \bigwedge_{x \in A} \big(A(y, x) \swarrow \mu(x)\big). \tag{2}$$

If $\mathbf{ub}(\mu)$ is represented by some $a \in A$, i.e.,

$$\mathbf{ub}(\mu)(y) = A(a, y), \tag{3}$$

we say that a is a *supremum* of μ, and denote it by $\sup\mu$. Dually, if $\mathbf{lb}(\mu)$ is represented by some $b\in A$, i.e.,

$$\mathbf{lb}(\mu)(y)=A(y,b), \tag{4}$$

we say that b is an *infimum* of μ, and denote it by $\inf\mu$.

Definition 3.7. An L-ordered set A is a *complete L-lattice* if every L-subset μ of A has both a supremum and an infimum.

In order to give a simple characterization of complete L-lattices, we introduce the following.

Definition 3.8. [14, 24] Let A be an L-ordered set.

(1) A is *tensored* if for all $a\in L$, $x\in A$, there is an element $a\otimes x\in A$ such that for all $y\in A$,

$$A(a\otimes x,y)=a\searrow A(x,y) \tag{5}$$

(2) A is *cotensored* if for all $a\in L$, $x\in A$, there is an element $a\rightarrowtail x\in A$ such that for all $y\in A$,

$$A(y,a\rightarrowtail x)=A(y,x)\swarrow a \tag{6}$$

Example 3.9.
(1) For all $x\in X$, $I\otimes x=I\rightarrowtail x=x$.
(2) If A is tensored, then A_0 has a bottom element $\bot_A$ and for all $x\in X$, $0\otimes x=\bot_A$.
(3) If A is cotensored, then A_0 has a top element $\top_A$ and for all $x\in X$, $0\rightarrowtail x=\top_A$.

Theorem 3.10. [24] *An L-ordered set A is a complete L-lattice if and only if*

(1) *A is tensored,*
(2) *A is cotensored,*
(3) *the underlying ordered set A_0 of A is a complete lattice.*

In this case, for each L-subset $\mu\in L^A$,

$$\sup\mu=\bigvee_{x\in A}\big(\mu(x)\otimes x\big);\quad \inf\mu=\bigwedge_{x\in A}\big(\mu(x)\rightarrowtail x\big),$$

where $\bigvee$ and $\bigwedge$ denote respectively the join and meet in the complete lattice A_0.

Example 3.11. For every set X, the L-powerset L^X and $(L_c^X)^{\mathrm{op}}$ are both complete L-lattices. For $a\in L,\mu\in L^X$,

$$a\otimes\mu=\mu\&a;\quad a\rightarrowtail\mu=\mu\swarrow a. \tag{7}$$

Hence, given an L-subset $\mathcal{U}:L^X\longrightarrow L$,

$$\sup\mathcal{U}=\bigvee_{\mu\in L^X}\big(\mu\&\mathcal{U}(\mu)\big),\quad \inf\mathcal{U}=\bigwedge_{\mu\in L^X}\big(\mu\swarrow\mathcal{U}(\mu)\big). \tag{8}$$

For $a\in L,\mu\in(L_c^X)^{\mathrm{op}}$,

$$a\otimes\mu=a\searrow\mu;\quad a\rightarrowtail\mu=a\&\mu. \tag{9}$$

Definition 3.12. [23] A pair of L-order preserving functions $f:A\longrightarrow B$ and $g:B\longrightarrow A$ is called an *L-adjunction* (denoted by $f\dashv g:A\rightharpoonup B$) if $B(f(x),y)=A(x,g(y))$ for all $x\in A,y\in B$. In this case, f is called a *left adjoint* of g and g a *right adjoint* of f.

Example 3.13. (1) For any sets X,Y and function $f:X\longrightarrow Y$, $f^{\rightarrow}\dashv f^{\leftarrow}:L^X\rightharpoonup L^Y$, where $f^{\rightarrow}:L^X\longrightarrow L^Y$ and $f^{\leftarrow}:L^Y\longrightarrow L^X$ are given by $f^{\rightarrow}(\mu)(y)=\bigvee_{f(x)=y}\mu(x)$ and $f^{\leftarrow}(\lambda)(x)=\lambda(f(x))$.
(2) For each L-ordered set A, $\mathbf{ub}\dashv\mathbf{lb}:L^A\rightharpoonup(L_c^A)^{\mathrm{op}}$.
(3) Let A be a complete L-lattice, then $\sup\dashv\mathbf{y}:L^A\rightharpoonup A$ and $\mathbf{y}'\dashv\inf:A\rightharpoonup(L_c^A)^{\mathrm{op}}$.
(4) Let A be an L-ordered set and $x\in A$, then $(-)\otimes x\dashv A(x,-):L\rightharpoonup A$ if A is tensored, and $A(-,x)\dashv(-)\rightarrowtail x:A\rightharpoonup L_c^{\mathrm{op}}$ if A is cotensored.

The following propositions are proved in [17] when L is a complete residuated lattice, i.e., a commutative unital quantale in which the unit I is equal to the top element 1. The proofs are similar when L is a general unital quantale.

Proposition 3.14. [17] *Let A, B be complete L-lattices and $f : A \longrightarrow B$ a function. Then the following are equivalent:*

1. *$f : A \longrightarrow B$ is L-order preserving;*
2. *$f : A_0 \longrightarrow B_0$ is order preserving and $a \otimes_B f(x) \leq f(a \otimes_A x)$ for all $a \in L$ and $x \in A$;*
3. *$f : A_0 \longrightarrow B_0$ is order preserving and $f(a \rightarrowtail_A x) \leq a \rightarrowtail_B f(x)$ for all $a \in L$ and $x \in A$.*

Proposition 3.15. [24] *Let A, B be complete L-lattices and $f : A \longrightarrow B$ a function. Then*

1. *$f : A \longrightarrow B$ is a left adjoint if and only if $f : A_0 \longrightarrow B_0$ preserves joins and f preserves tensors in the sense that $f(a \otimes_A x) = a \otimes_B f(x)$ for all $a \in L$ and $x \in A$.*
2. *$f : A \longrightarrow B$ is a right adjoint if and only if $f : A_0 \longrightarrow B_0$ preserves meets and f preserves cotensors in the sense that $f(a \rightarrowtail_A x) = a \rightarrowtail_B f(x)$ for all $a \in L$ and $x \in A$.*

Let A be a complete L-lattice and $x \in A$. Since $A(x, -) : A \longrightarrow L$ is a right adjoint and $A(-, x) : A \longrightarrow L_c^{\rm op}$ is a left adjoint, it holds that for each subset $\{y_j : j \in J\} \subseteq A$,

$$A\Big(x, \bigwedge_{j\in J} y_j\Big) = \bigwedge_{j\in J} A(x, y_j); \quad A\Big(\bigvee_{j\in J} y_j, x\Big) = \bigwedge_{j\in J} A(y_j, x), \tag{10}$$

where $\bigvee$ and $\bigwedge$ denote the join and the meet in A_0 respectively.

Corollary 3.16. *Let A, B be complete L-lattices and $f : A \longrightarrow B$ a function.*

1. *$f : A \longrightarrow B$ is a left adjoint if and only if f preserves supremum in the sense that $f(\sup_A \mu) = \sup_B f^{\rightarrow}(\mu)$ for all $\mu \in L^A$.*
2. *$f : A \longrightarrow B$ is a right adjoint if and only if f preserves infimum in the sense that $f(\inf_A \mu) = \inf_B f^{\rightarrow}(\mu)$ for all $\mu \in L^A$.*

The category of complete L-lattices and suprema-preserving maps (i.e., left adjoints) is denoted by L-**Sup**.

3.3. L-closure spaces.

The notions of L-closure operators and L-closure systems in [5, 17] can be extended in a straightforward way to the setting that L is a unital quantale.

Definition 3.17. Let A be an L-ordered set. An L-order preserving function $c : A \longrightarrow A$ is called an *L-closure operator* if

- (C1) for all $x \in A$, $A(x, c(x)) \geq I$, i.e. $x \leq c(x)$ in the underlying ordered set A_0 of A;
- (C2) $c \circ c = c$.

Example 3.18. Let A, B be L-ordered sets and $f \dashv g : A \rightharpoonup B$ be an L-adjunction. Then $g \circ f : A \longrightarrow A$ is an L-closure operator.

Definition 3.19. Let A be a complete L-lattice with cotensor $\rightarrowtail$. A subset $C \subseteq A$ is an *L-closure system* of A if

- (i) for every subset $\{x_t\}_{t\in T} \subseteq C$, the meet $\bigwedge_{t\in T} x_t$ of $\{x_t\}_{t\in T}$ in A_0 belongs to C;
- (ii) for all $x \in C$ and $a \in L$, the cotensor $a \rightarrowtail x$ belongs to C.

L-closure operators and L-closure systems are connected by the following proposition, and the proof is similar to the case that L is a complete residuated lattice [17].

Proposition 3.20. *Let A be a complete L-lattice, C a subset of A. The following are equivalent:*

1. *C is an L-closure system of A.*
2. *The inclusion function $i : C \longrightarrow A$ is a right adjoint.*
3. *There is an L-closure operator $c : A \longrightarrow A$ such that $C = c(A)$.*

The above proposition establishes a bijection between L-closure operators and L-closure systems. Precisely, for an L-closure operator $c : A \longrightarrow A$, the fixed points $c(A)$ is an L-closure system of A. Conversely, given an L-closure system $C \subseteq A$, $c = i \circ h$ is an L-closure operator on A, where h is the left adjoint of the inclusion $i : C \longrightarrow A$. In particular, for an L-closure operator $c : A \longrightarrow A$, we have an L-adjunction $c \dashv i : A \rightharpoonup c(A)$, where i is the inclusion.

Let A be a complete L-lattice with cotensor $\rightarrowtail$. Then the fixed points $c(A)$ of an L-closure operator $c : A \longrightarrow A$ is closed under the formation of cotensors and meets in A_0, and $c(A)$ is itself a complete L-lattice. For an L-subset $\mu : c(A) \longrightarrow L$, the infimum of μ in $c(A)$ is the same as the infimum of μ in A (i.e., $\inf_{c(A)}\mu = \bigwedge_{x\in c(A)} \mu(x) \rightarrowtail x$) and the supremum of μ in $c(A)$ is given by $\sup_{c(A)} \mu = c(\sup_A \mu)$.

Definition 3.21. An *L-closure space* is a pair (X, c), where X is a set and $c : L^X \longrightarrow L^X$ is an L-closure operator. A continuous function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces is a function $f : X \longrightarrow Y$ such that $f^{\rightarrow} \circ c(\mu) \leq d \circ f^{\rightarrow}(\mu)$ for all $\mu \in L^X$.

The category of L-closure spaces and continuous functions is denoted by L-**Cls**.

4. An adjunction between complete L-lattices and L-closure spaces

Let (X, c) be an L-closure space. An L-subset $\mu \in L^X$ is said to be closed if $c(\mu) = \mu$. Since L^X is a complete L-lattice, it follows that all the closed L-subsets of (X, c) form a complete L-lattice $c(L^X)$.

Lemma 4.1. *A function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces is continuous if and only if $f^{\leftarrow}(\lambda) \in c(L^X)$ whenever $\lambda \in d(L^Y)$.*

Proof. Suppose that f is continuous and $\lambda \in d(L^Y)$, then

$$f^{\rightarrow} \circ c \circ f^{\leftarrow}(\lambda) \leq d \circ f^{\rightarrow} \circ f^{\leftarrow}(\lambda) \leq d(\lambda) = \lambda.$$

Consequently $c \circ f^{\leftarrow}(\lambda) \leq f^{\leftarrow}(\lambda)$ and $f^{\leftarrow}(\lambda) \in c(L^X)$.

Conversely, suppose that $f^{\leftarrow}(\lambda) \in c(L^X)$ for all $\lambda \in d(L^Y)$. Then for each $\mu \in L^X$, $f^{\leftarrow} \circ d \circ f^{\rightarrow}(\mu) \in c(L^X)$. Since $\mu \leq f^{\leftarrow} \circ f^{\rightarrow}(\mu) \leq f^{\leftarrow} \circ d \circ f^{\rightarrow}(\mu)$, it follows that $c(\mu) \leq f^{\leftarrow} \circ d \circ f^{\rightarrow}(\mu)$, hence $f^{\rightarrow} \circ c(\mu) \leq d \circ f^{\rightarrow}(\mu)$. □

If L is a completely distributive lattice (as required in [29]), then it is easy to see that the L-closure spaces considered in this paper are a special kind of L-closure spaces in [29]. In this case, the above lemma can also be obtained from Theorem 5.1 in [29] as pointed out by a reviewer.

Given a continuous function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces, define a pair of functions

$$f_* : c(L^X) \longrightarrow d(L^Y), \quad f^* : d(L^Y) \longrightarrow c(L^X)$$

by

$$f_*(\mu) = d \circ f^{\rightarrow}(\mu), \quad f^*(\lambda) = f^{\leftarrow}(\lambda)$$

for all $\mu \in c(L^X)$ and $\lambda \in d(L^Y)$.

Proposition 4.2. *If $f : (X, c) \longrightarrow (Y, d)$ is a continuous function between L-closure spaces, then $f_* \dashv f^* : c(L^X) \rightharpoonup d(L^Y)$.*

Proof. Since $f^{\rightarrow} \dashv f^{\leftarrow} : L^X \rightharpoonup L^Y$, it remains to show that $L^Y\big(d \circ f^{\rightarrow}(\mu), \lambda\big) = L^Y\big(f^{\rightarrow}(\mu), \lambda\big)$ for all $\mu \in c(L^X)$ and $\lambda \in d(L^Y)$. Indeed, $L^Y\big(f^{\rightarrow}(\mu), \lambda\big) \leq L^Y\big(d \circ f^{\rightarrow}(\mu), \lambda\big)$ because d is L-order preserving, and the reverse inequality follows from (10) and the fact that d is an L-closure operator. □

The above proposition gives rise to a functor

$$T : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Sup}$$

which maps a continuous function $f : (X, c) \longrightarrow (Y, d)$ to $f_* : c(L^X) \longrightarrow d(L^Y)$.

Let A be a complete L-lattice. Then $c_A = \mathbf{y} \circ \sup : L^A \longrightarrow A \longrightarrow L^A$ is an L-closure operator, hence (A, c_A) is an L-closure space.

Proposition 4.3. *Let A, B be complete L-lattices and $f : A \longrightarrow B$ be a left adjoint. Then $f : (A, c_A) \longrightarrow (B, c_B)$ is a continuous function.*

Proof. Since f preserves supremum (Corollary 3.16), it holds that

$$\begin{aligned} f^{\rightarrow} \circ c_A(\mu)(b) &= \bigvee_{f(a)=b} c_A(\mu)(a) \\ &= \bigvee_{f(a)=b} A(a, \sup \mu) \\ &\leq \bigvee_{f(a)=b} B\big(f(a), f(\sup \mu)\big) \\ &= B\big(b, \sup f^{\rightarrow}(\mu)\big) \\ &= c_B \circ f^{\rightarrow}(\mu)(b) \end{aligned}$$

for all $\mu \in L^A$ and $b \in B$, hence $f : (A, c_A) \longrightarrow (B, c_B)$ is continuous. □

The above proposition gives a functor $D : L$-**Sup** $\longrightarrow L$-**Cls**.

Proposition 4.4. *The functor $T \circ D$ is naturally isomorphic to the identity functor on L-***Sup***. In particular, A is isomorphic to $T \circ D(A)$ for each complete L-lattice A.*

Proof. For each $a \in A$, since $\mathbf{y}(a) = \mathbf{y} \circ \sup \circ \mathbf{y}(a) = c_A \circ \mathbf{y}(a)$, we get that $\mathbf{y}(a)$ is a closed L-subset of (A, c_A). Then it follows from the Yoneda lemma that the correspondence $a \mapsto \mathbf{y}(a)$ is an L-isometry $\nu_A : A \longrightarrow T \circ D(A) (= c_A(L^A))$. It is clear that ν_A is surjective, hence an isomorphism. We leave it to the reader to check that $\{\nu_A\}$ is a natural isomorphism from the identity functor on L-**Sup** to $T \circ D$. □

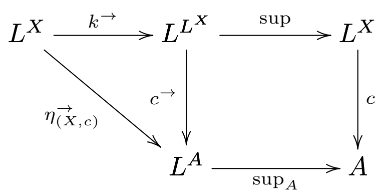
Theorem 4.5. *$T : L$-***Cls*** $\longrightarrow L$-***Sup*** is a left adjoint of $D : L$-***Sup*** $\longrightarrow L$-***Cls***.*

Proof. Given an L-closure space (X, c), let A be the complete L-lattice $c(L^X)$ of all the closed L-subsets of X. Then $D \circ T(X, c) = (A, c_A)$. Define $\eta_{(X,c)} : X \longrightarrow A$ by $\eta_{(X,c)}(x) = c(I_x)$, where $I_x \in L^X$ is given by $I_x(x) = I$ and $I_x(y) = 0$ whenever $y \neq x$.

In the following we show that $\eta = \{\eta_{(X,c)}\}$ is a natural transformation from the identity functor to $D \circ T$ and it is the unit of the desired adjunction.

Step 1. $\eta_{(X,c)} : (X, c) \longrightarrow (A, c_A)$ is continuous, i.e. $\eta^{\rightarrow}_{(X,c)} \circ c(\mu) \leq c_A \circ \eta^{\rightarrow}_{(X,c)}(\mu)$ for all $\mu \in L^X$.

Firstly, we show that $c(\mu) = \sup_A \circ \eta^{\rightarrow}_{(X,c)}(\mu)$ for all $\mu \in L^X$. Consider the diagram



where $k : X \longrightarrow L^X$ is given by $k(x) = I_x$ for all $x \in X$, and sup is the supremum operation in the complete L-lattice L^X. The commutativity of the left triangle follows from $\eta_{(X,c)} = c \circ k$. Since $c : L^X \longrightarrow A$ is a left adjoint, it preserves supremum, thus the right square commutes. The whole diagram is then commutative. For each $\mu \in L^X$, we have that

$$\sup \circ k^{\rightarrow}(\mu) = \bigvee_{\lambda \in L^X} \lambda \& k^{\rightarrow}(\mu)(\lambda) = \bigvee_{x \in X} I_x \& \mu(x) = \mu.$$

Consequently, $c(\mu) = \sup_A \circ \eta^{\rightarrow}_{(X,c)}(\mu)$ for all $\mu \in L^X$.

Secondly, we show that $\eta^{\rightarrow}_{(X,c)}(\mu) \leq \mathbf{y}_A(\mu)$ for each closed L-subset μ in (X, c). For each $\lambda \in A$, we have that $\eta^{\rightarrow}_{(X,c)}(\mu)(\lambda) = \bigvee_{\eta_{(X,c)}(x) = \lambda} \mu(x)$ by definition. If $\lambda = \eta_{(X,c)}(x) = c(I_x)$, then

$$\mu(x) = L^X(I_x, \mu) \leq A(\lambda, \mu) = \mathbf{y}_A(\mu)(\lambda)$$

since c is L-order preserving. Consequently, $\eta^{\rightarrow}_{(X,c)}(\mu) \leq \mathbf{y}_A(\mu)$.

Therefore,

$$\eta^{\rightarrow}_{(X,c)} \circ c(\mu) \leq \mathbf{y}_A \circ \sup_A \circ \eta^{\rightarrow}_{(X,c)}(\mu) = c_A \circ \eta^{\rightarrow}_{(X,c)}(\mu),$$

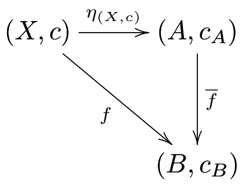
as desired.

Step 2. $\eta = \{\eta_{(X,c)}\}$ is a natural transformation. Let $f : (X, c) \longrightarrow (Y, d)$ be a continuous function, we must show that for all $x \in X$,

$$d(I_{f(x)}) = \eta_{(Y,d)} \circ f(x) = (D \circ T)(f) \circ \eta_{(X,c)}(x) = d \circ f^{\rightarrow}(c(I_x)).$$

On one hand, $I_{f(x)} = f^{\rightarrow}(I_x) \leq f^{\rightarrow}(c(I_x))$ leads to $d(I_{f(x)}) \leq d \circ f^{\rightarrow}(c(I_x))$. On the other hand, since f is continuous, $f^{\rightarrow}(c(I_x)) \leq d \circ f^{\rightarrow}(I_x) = d(I_{f(x)})$, hence $d \circ f^{\rightarrow}(c(I_x)) \leq d(I_{f(x)})$.

Step 3. $\eta_{(X,c)} : (X, c) \longrightarrow (A, c_A)$ is universal in the sense that for any complete L-lattice B and continuous function $f : (X, c) \longrightarrow (B, c_B)$, there exists a unique suprema-preserving map $\overline{f} : A \longrightarrow B$ that makes the following diagram commute:



Existence. For all $\mu \in A$, let $\overline{f}(\mu) = \sup_B \circ f^{\rightarrow}(\mu) = \bigvee_{x \in X} \mu(x) \otimes_B f(x)$. In other words, $\overline{f} : A \longrightarrow B$ is the following composition of L-order preserving functions

$$A \hookrightarrow L^X \xrightarrow{f^{\rightarrow}} L^B \xrightarrow{\sup_B} B.$$

We assert that $\overline{f}$ satisfies the requirement.

Firstly, we show that $\overline{f} : A \longrightarrow B$ is a left adjoint. Indeed, $\overline{f}$ has a right adjoint $g : B \longrightarrow A$ given by $g(b) = f^*(\mathbf{y}_B(b))$. g is well-defined since $\mathbf{y}_B(b)$ is a closed L-subset in (B, c_B) for each $b \in B$ by Proposition 4.4. For all $\mu \in A$ and $b \in B$,

$$\begin{aligned}
B\big(\overline{f}(\mu), b\big) &= \bigwedge_{y \in B} \Big(f^{\rightarrow}(\mu)(y) \searrow B(y, b)\Big) \quad \text{(by (3))} \\
&= \bigwedge_{y \in B} \Big(\bigvee_{f(x)=y} \mu(x) \searrow B(y, b)\Big) \\
&= \bigwedge_{y \in B} \bigwedge_{f(x)=y} \Big(\mu(x) \searrow B(y, b)\Big) \\
&= \bigwedge_{x \in X} \Big(\mu(x) \searrow B(f(x), b)\Big) \\
&= \bigwedge_{x \in X} \Big(\mu(x) \searrow \mathbf{y}_B(b)(f(x))\Big) \\
&= L^X\big(\mu, f^* \circ \mathbf{y}_B(b)\big) \\
&= A\big(\mu, g(b)\big).
\end{aligned}$$

Hence $\overline{f} \dashv g : A \rightharpoonup B$ and $\overline{f}$ is a left adjoint.

Secondly, we show that $\overline{f} \circ \eta_{(X,c)}(x) = f(x)$ for all $x \in X$. Since $f : (X, c) \longrightarrow (B, c_B)$ is continuous, it follows that

$$f^{\rightarrow} \circ c(I_x) \leq c_B \circ f^{\rightarrow}(I_x) = \mathbf{y}_B \circ \sup_B I_{f(x)} = \mathbf{y}_B(f(x))$$

for all $x \in X$. Hence,

$$c(I_x)(y) \leq f^{\leftarrow} \circ \mathbf{y}_B(f(x))(y) = B(f(y), f(x))$$

for all $x, y \in X$. Then

$$c(I_x)(y) \& B(f(x), b) \leq B(f(y), f(x)) \& B(f(x), b) \leq B(f(y), b)$$

for all $x, y \in X$ and $b \in B$. Therefore,

$$\begin{aligned} B(f(x),b) &= \bigwedge_{y\in X} \Big(c(I_x)(y) \searrow B(f(y),b)\Big) \\ &= L^X\big(c(I_x), f^{\leftarrow}\circ \mathbf{y}_B(b)\big) \\ &= L^B\big(f^{\rightarrow}\circ c(I_x), \mathbf{y}_B(b)\big) \\ &= B\big(\sup_B \circ f^{\rightarrow}\circ c(I_x), b\big) \quad \text{(by (3))} \\ &= B\big(\overline{f}\circ \eta_{(X,c)}(x), b\big) \end{aligned}$$

for all $x \in X$ and $b \in B$, and consequently $\overline{f}\circ\eta_{(X,c)} = f$.

Uniqueness. Suppose $h : A \longrightarrow B$ is a left adjoint that makes the diagram commute. For each $\mu \in A$, since

$$\mu = c(\mu) = c\Big(\bigvee_{x\in X} I_x \& \mu(x)\Big) = \bigvee_{x\in X} \mu(x) \otimes_A c(I_x),$$

it follows that

$$\begin{aligned} h(\mu) &= h\Big(\bigvee_{x\in X} \mu(x)\otimes_A c(I_x)\Big) \\ &= \bigvee_{x\in X} \mu(x) \otimes_B \big(h\circ\eta_{(X,c)}(x)\big) \\ &= \bigvee_{x\in X} \mu(x)\otimes_B f(x) \\ &= \overline{f}(\mu). \end{aligned}$$

Therefore, $h = \overline{f}$. □

5. An adjunction between L-contexts and L-closure spaces

Let (X, Y, R) be an L-context, define a pair of operators

$$R_{\uparrow} : L^X \longrightarrow L^Y \quad \text{and} \quad R^{\downarrow} : L^Y \longrightarrow L^X$$

by

$$R_{\uparrow}(\mu)(y) = \bigwedge_{x\in X} \big(\mu(x) \searrow R(x,y)\big) \quad \text{and} \quad R^{\downarrow}(\lambda)(x) = \bigwedge_{y\in Y} \big(R(x,y) \swarrow \lambda(y)\big).$$

Then it is easy to see that

$$(L^Y_c)^{\mathrm{op}}\big(R_{\uparrow}(\mu), \lambda\big) = L^X\big(\mu, R^{\downarrow}(\lambda)\big)$$

for all $\mu \in L^X$ and $\lambda \in L^Y$. Hence

$$R_{\uparrow} \dashv R^{\downarrow} : L^X \rightharpoonup (L^Y_c)^{\mathrm{op}}$$

is an L-adjunction[2]. Therefore, $R^{\downarrow}\circ R_{\uparrow} : L^X \longrightarrow L^X$ is an L-closure operator.

Proposition 5.1. *Let $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ be an infomorphism between L-contexts. Then $f : (X, R^{\downarrow}\circ R_{\uparrow}) \longrightarrow (A, S^{\downarrow}\circ S_{\uparrow})$ is continuous.*

Proof. Consider the following diagram:

(11)
$$\begin{array}{ccccc} L^X & \xrightarrow{R_{\uparrow}} & L^Y & \xrightarrow{R^{\downarrow}} & L^X \\ {\scriptstyle f^{\rightarrow}}\downarrow & & \downarrow{\scriptstyle g^{\leftarrow}} & & \downarrow{\scriptstyle f^{\rightarrow}} \\ L^A & \xrightarrow[S_{\uparrow}]{} & L^B & \xrightarrow[S^{\downarrow}]{} & L^A \end{array}$$

[2] When L is a complete residuated lattice, the pair $(R_{\uparrow}, R^{\downarrow})$ is introduced by Bělohlávek first and is called a fuzzy Galois connection [4, 6].

We must show that $f^{\rightarrow}\circ R^{\downarrow}\circ R_{\uparrow}(\mu)\leq S^{\downarrow}\circ S_{\uparrow}\circ f^{\rightarrow}(\mu)$ for all $\mu\in L^X$. The conclusion is contained in the following two lemmas. □

Lemma 5.2. *The left square of (11) commutes if and only if $(f,g):(X,Y,R)\longrightarrow(A,B,S)$ is an infomorphism.*

Proof. Necessity is easy since for all $x\in X, b\in B$,

$$R(x,g(b))=R_{\uparrow}(I_x)(g(b))=g^{\leftarrow}\big(R_{\uparrow}(I_x)\big)(b)=S_{\uparrow}\big(f^{\rightarrow}(I_x)\big)(b)=S_{\uparrow}(I_{f(x)})(b)=S(f(x),b).$$

As for sufficiency, it holds that

$$\begin{aligned} g^{\leftarrow}\circ R_{\uparrow}(\mu)(b) &= R_{\uparrow}(\mu)(g(b)) \\ &= \bigwedge_{x\in X}\Big(\mu(x)\searrow R(x,g(b))\Big) \\ &= \bigwedge_{x\in X}\Big(\mu(x)\searrow S(f(x),b)\Big) \\ &= \bigwedge_{a\in A}\Big(\bigvee_{f(x)=a}\mu(x)\searrow S(a,b)\Big) \\ &= \bigwedge_{a\in A}\Big(f^{\rightarrow}(\mu)(a)\searrow S(a,b)\Big) \\ &= S_{\uparrow}\circ f^{\rightarrow}(\mu)(b) \end{aligned}$$

for all $\mu\in L^X$ and $b\in B$, hence the conclusion follows. □

Lemma 5.3. *Consider the right square of (11). Then $f^{\rightarrow}\circ R^{\downarrow}\leq S^{\downarrow}\circ g^{\leftarrow}$ if and only if $R(x,g(b))\leq S(f(x),b)$ for all $x\in X, b\in B$.*

Proof. Necessity is easy since for all $x\in X, b\in B$,

$$R(x,g(b))=R^{\downarrow}(I_{g(b)})(x)\leq f^{\rightarrow}\circ R^{\downarrow}(I_{g(b)})(f(x))\leq S^{\downarrow}\circ g^{\leftarrow}(I_{g(b)})(f(x))\leq S(f(x),b).$$

As for sufficiency, for all $\lambda\in L^Y$ and $a\in A$, we have that

$$\begin{aligned} f^{\rightarrow}\circ R^{\downarrow}(\lambda)(a) &= \bigvee_{f(x)=a}R^{\downarrow}(\lambda)(x) \\ &= \bigvee_{f(x)=a}\bigwedge_{y\in Y}\Big(R(x,y)\swarrow\lambda(y)\Big) \\ &\leq \bigvee_{f(x)=a}\bigwedge_{b\in B}\Big(R(x,g(b))\swarrow\lambda(g(b))\Big) \\ &\leq \bigvee_{f(x)=a}\bigwedge_{b\in B}\Big(S(f(x),b)\swarrow\lambda(g(b))\Big) \\ &= \bigwedge_{b\in B}\Big(S(a,b)\swarrow\lambda(g(b))\Big) \\ &= \bigwedge_{b\in B}\Big(S(a,b)\swarrow g^{\leftarrow}(\lambda)(b)\Big) \\ &= S^{\downarrow}\circ g^{\leftarrow}(\lambda)(a), \end{aligned}$$

thus, the conclusion follows. □

By virtue of Proposition 5.1 we obtain a functor $U:L\text{-}\mathbf{Ctx}\longrightarrow L\text{-}\mathbf{Cls}$ that sends an infomorphism

$$(f,g):(X,Y,R)\longrightarrow(A,B,S)$$

to the continuous function

$$f:(X,R^{\downarrow}\circ R_{\uparrow})\longrightarrow(A,S^{\downarrow}\circ S_{\uparrow}).$$

Given a continuous function $f : (X, c) \longrightarrow (Y, d)$ between L-closure spaces, let $c(L^X)$ and $d(L^Y)$ denote respectively the sets of all closed L-subsets of X and Y. Consider the function $f^* : d(L^Y) \longrightarrow c(L^X)$ that sends each closed L-subset λ of Y to $f^*(\lambda) = f^{\leftarrow}(\lambda)$. Then we have the following

Proposition 5.4. *$(f, f^*) : (X, c(L^X), R) \longrightarrow (Y, d(L^Y), S)$ is an infomorphism, where $R(x, \mu) = \mu(x)$ for all $x \in X$ and $\mu \in c(L^X)$, $S(y, \lambda) = \lambda(y)$ for all $y \in Y$ and $\lambda \in d(L^Y)$.*

Proof. For all $x \in X$ and $\lambda \in d(L^Y)$, we have that

$$R(x, f^*(\lambda)) = f^*(\lambda)(x) = f^{\leftarrow}(\lambda)(x) = \lambda(f(x)) = S(f(x), \lambda),$$

the conclusion thus follows. □

From the above proposition we obtain a functor

$$F : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Ctx}$$

that sends a continuous function $f : (X, c) \longrightarrow (Y, d)$ to the infomorphism

$$(f, f^*) : (X, c(L^X), R) \longrightarrow (Y, d(L^Y), S).$$

Lemma 5.5. *For each L-closure space (X, c), $U \circ F(X, c) = (X, c)$.*

Proof. By definition of the functor F, $F(X, c)$ is the L-context $(X, c(L^X), R)$, where $R(x, \mu) = \mu(x)$ for all $x \in X$ and $\mu \in c(L^X)$. We must show that $c = R^{\downarrow} \circ R_{\uparrow}$. Since $R^{\downarrow} \circ R_{\uparrow}$ is an L-closure operator, it remains to show that for any $\mu \in L^X$, $c(\mu) = \mu$ if and only if $R^{\downarrow} \circ R_{\uparrow}(\mu) = \mu$.

If $c(\mu) = \mu$, then

$$\begin{aligned} R^{\downarrow} \circ R_{\uparrow}(\mu)(x) &= \bigwedge_{\phi \in c(L^X)} \Big(R(x, \phi) \swarrow \big(\bigwedge_{z \in X} \mu(z) \searrow R(z, \phi) \big) \Big) \\ &= \bigwedge_{\phi \in c(L^X)} \Big(\phi(x) \swarrow \big(\bigwedge_{z \in X} \mu(z) \searrow \phi(z) \big) \Big) \\ &\leq \mu(x) \swarrow \Big(\bigwedge_{z \in X} \mu(z) \searrow \mu(z) \Big) \\ &\leq \mu(x) \end{aligned}$$

for all $x \in X$, hence $\mu \geq R^{\downarrow} \circ R_{\uparrow}(\mu)$. The converse inequality $\mu \leq R^{\downarrow} \circ R_{\uparrow}(\mu)$ is trivial since $R^{\downarrow} \circ R_{\uparrow}$ is an L-closure operator. Therefore, $\mu = R^{\downarrow} \circ R_{\uparrow}(\mu)$.

Suppose that $R^{\downarrow} \circ R_{\uparrow}(\mu) = \mu$. Then

$$\mu = R^{\downarrow}(R_{\uparrow}(\mu)) = \bigwedge_{\phi \in c(L^X)} \big(\phi \swarrow R_{\uparrow}(\mu)(\phi) \big).$$

Since $c(L^X)$ is closed with respect to meets and cotensors in L^X, we obtain that

$$\bigwedge_{\phi \in c(L^X)} \big(\phi \swarrow R_{\uparrow}(\mu)(\phi) \big) \in c(L^X),$$

hence $\mu \in c(L^X)$ and $c(\mu) = \mu$. □

Theorem 5.6. *$F : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Ctx}$ is a left adjoint of $U : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Cls}$.*

Proof. For each L-closure space (X, c), $\mathrm{id}_X : (X, c) \longrightarrow U \circ F(X, c)$ is clearly continuous and $\{\mathrm{id}_X\}$ is a natural transformation from the identity functor to $U \circ F$ by the above lemma. Then it remains to show that for each L-context (A, B, R) and continuous function $f : (X, c) \longrightarrow (A, R^{\downarrow} \circ R_{\uparrow})$, there is a unique infomorphism

$$(h, g) : F(X, c) \longrightarrow (A, B, R)$$

such that the diagram

$$\begin{array}{ccc} (X,c) & \xrightarrow{\mathrm{id}_X} & U\circ F(X,c) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle U(h,g)} \\ & & U(A,B,R) \end{array}$$

is commutative. By definition, $F(X,c) = (X, c(L^X), S)$ and $U(h,g) = h$, where $S(x,\mu) = \mu(x)$. Thus, we only need to show that there is a unique function $g : B \longrightarrow c(L^X)$ such that

$$(f,g) : (X, c(L^X), S) \longrightarrow (A,B,R)$$

is an infomorphism.

Let $g(b) = f^{\leftarrow}(R^{\downarrow}(I_b))$ for each $b \in B$. Since $R^{\downarrow} \circ R_{\uparrow} \circ R^{\downarrow}(I_b) = R^{\downarrow}(I_b)$, it follows that $R^{\downarrow}(I_b)$ is a closed L-subset in $(A, R^{\downarrow} \circ R_{\uparrow})$, hence $f^{\leftarrow}(R^{\downarrow}(I_b)) \in c(L^X)$ by the continuity of $f : (X,c) \longrightarrow (A, R^{\downarrow} \circ R_{\uparrow})$. This shows that $g : B \longrightarrow c(L^X)$ is well-defined. Now we check that $(f,g) : (X, c(L^X), S) \longrightarrow (A,B,R)$ is an infomorphism. This is easy since

$$S(x, g(b)) = g(b)(x) = R^{\downarrow}(I_b)(f(x)) = R(f(x), b)$$

for all $x \in X$ and $b \in B$. This proves the existence of g.

To see the uniqueness, suppose that $g' : B \longrightarrow c(L^X)$ is a function such that

$$(f,g') : (X, c(L^X), S) \longrightarrow (A,B,R)$$

is an infomorphism. Then for all $b \in B$ and $x \in X$,

$$g'(b)(x) = S(x, g'(b)) = R(f(x), b) = R^{\downarrow}(I_b)(f(x)) = g(b)(x),$$

hence $g' = g$. □

As an immediate corollary of Theorem 5.6 and Lemma 5.5 we obtain that the category L-**Cls** is a coreflective subcategory of L-**Ctx**.

6. The concept lattice functors

6.1. The formal concept lattice functor.

Given an L-context (X,Y,R), a *formal concept* of (X,Y,R) is a pair $(\mu,\lambda) \in L^X \times L^Y$ such that $\lambda = R_{\uparrow}(\mu)$ and $\mu = R^{\downarrow}(\lambda)$[3]. μ is called the *extent* and λ is called the *intent*. The set of all formal concepts of (X,Y,R) is denoted by $\mathfrak{B}(X,Y,R)$.

For $(\mu_1,\lambda_1), (\mu_2,\lambda_2) \in \mathfrak{B}(X,Y,R)$, let

$$\mathfrak{B}(X,Y,R)\big((\mu_1,\lambda_1),(\mu_2,\lambda_2)\big) = L^X(\mu_1,\mu_2) = (L_c^Y)^{\mathrm{op}}(\lambda_1,\lambda_2). \tag{12}$$

Then $\mathfrak{B}(X,Y,R)$ becomes an L-ordered set. The projection

$$\pi_1 : \mathfrak{B}(X,Y,R) \longrightarrow L^X, \quad (\mu,\lambda) \mapsto \mu$$

is clearly an L-isometry. Since the image of π_1 is exactly the set of fixed points of the L-closure operator $R^{\downarrow} \circ R_{\uparrow} : L^X \longrightarrow L^X$, we obtain that $\mathfrak{B}(X,Y,R)$ is isomorphic to the L-closure system $R^{\downarrow} \circ R_{\uparrow}(L^X)$ of the complete L-lattice L^X, hence $\mathfrak{B}(X,Y,R)$ is itself a complete L-lattice, called the *formal concept lattice* of (X,Y,R).

Similarly, the projection

$$\pi_2 : \mathfrak{B}(X,Y,R) \longrightarrow (L_c^Y)^{\mathrm{op}}, \quad (\mu,\lambda) \mapsto \lambda$$

is also an L-isometry and the image of π_2 is exactly the set of fixed points of the function $R_{\uparrow} \circ R^{\downarrow} : L^Y \longrightarrow L^Y$. Hence $\mathfrak{B}(X,Y,R)$ is also isomorphic to $R_{\uparrow} \circ R^{\downarrow}(L^Y)$ endowed with L-order inherited from $(L_c^Y)^{\mathrm{op}}$ (which will be denoted by $R_{\uparrow} \circ R^{\downarrow}(L_c^Y)^{\mathrm{op}}$ in the sequel). Equation (12) shows that both

$$R_{\uparrow} : R^{\downarrow} \circ R_{\uparrow}(L^X) \longrightarrow R_{\uparrow} \circ R^{\downarrow}(L_c^Y)^{\mathrm{op}}$$

[3]It should be noted that this definition of formal concepts of L-contexts is different from that in [12, 21] in the non-commutative case.

and

$$R^{\downarrow} : R_{\uparrow} \circ R^{\downarrow}(L_c^Y)^{\mathrm{op}} \longrightarrow R^{\downarrow} \circ R_{\uparrow}(L^X)$$

are isomorphisms and are inverse to each other.

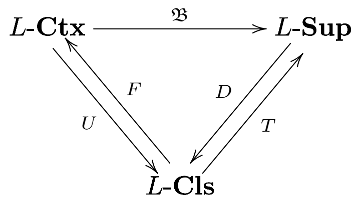
If we identify $\mathfrak{B}(X, Y, R)$ with $R^{\downarrow} \circ R_{\uparrow}(L^X)$, then the formal concept lattice $\mathfrak{B}(X, Y, R)$ is the set of the closed L-subsets of the L-closure space $(X, R^{\downarrow} \circ R_{\uparrow})$, i.e., $\mathfrak{B}(X, Y, R) = T \circ U(X, Y, R)$. Since both T and U are functorial, the correspondence

$$(X, Y, R) \mapsto \mathfrak{B}(X, Y, R)$$

defines a functor

$$\mathfrak{B} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup},$$

which is called the *formal concept lattice functor* (or, the concept lattice functor based on formal concept analysis). Note that the functor $\mathfrak{B}$ is the composition of a right adjoint functor U and a left adjoint functor T.



The formal concept lattice functor $\mathfrak{B} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$ sends an infomorphism

$$(f, g) : (X, Y, R) \longrightarrow (A, B, S)$$

to the left adjoint

$$f_* : R^{\downarrow} \circ R_{\uparrow}(L^X) \longrightarrow S^{\downarrow} \circ S_{\uparrow}(L^A)$$

between complete L-lattices.

Since F is a right inverse of U (Lemma 5.5) and $A \cong T \circ D(A)$ for each complete L-lattice A (Proposition 4.4), the fundamental theorem [7] of formal concept lattices can be immediately extended to the non-commutative fuzzy setting as follows.

Theorem 6.1. ([6] for the commutative case) *$\mathfrak{B}(X, Y, R)$ is a complete L-lattice for each L-context (X, Y, R) and each complete L-lattice is isomorphic to $\mathfrak{B}(X, Y, R)$ for some L-context (X, Y, R).*

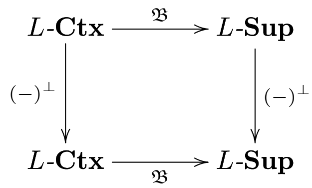
Definition 6.2. A category $\mathbf{C}$ is called *self dual* if there exists a contravariant functor $(-)^{\perp}$ on $\mathbf{C}$ such that $(-)^{\perp} \circ (-)^{\perp} = \mathrm{id}$, the identity functor on $\mathbf{C}$. For each morphism $f : A \longrightarrow B$, $f^{\perp} : B^{\perp} \longrightarrow A^{\perp}$ is called the *dual* of f.

L-**Ctx** is a self dual category. The dual of an infomorphism $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ is given by $(f, g)^{\perp} = (g, f) : (B, A, S^{\mathrm{op}}) \longrightarrow (Y, X, R^{\mathrm{op}})$.

L-**Sup** is a self dual category if $(L, \&)$ is a commutative unital quantale. For a left adjoint $f : A \longrightarrow B$ between complete L-lattices, let $g : B \longrightarrow A$ be the right adjoint of f. The dual $f^{\perp}$ of f is given by the left adjoint $g : B^{\mathrm{op}} \longrightarrow A^{\mathrm{op}}$.

The following proposition says that the formal concept lattice functor $\mathfrak{B} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$ preserves the dual structures of L-**Ctx** and L-**Sup** up to a natural isomorphism.

Proposition 6.3. *If $(L, \&)$ is a commutative unital quantale, then the diagram*



commutes up to a natural isomorphism.

Proof. By definition, the functors $(-)^{\perp}\circ\mathfrak{B}$ and $\mathfrak{B}\circ(-)^{\perp}$ map an L-context (X,Y,R) to $\mathfrak{B}(X,Y,R)^{\perp} = \big(R^{\downarrow}\circ R_{\uparrow}(L^X)\big)^{\mathrm{op}}$ and $\mathfrak{B}(Y,X,R^{\mathrm{op}}) = R_{\uparrow}\circ R^{\downarrow}(L^Y)$ respectively, hence

$$\eta_{(X,Y,R)} = R_{\uparrow} : \mathfrak{B}(X,Y,R)^{\perp} \longrightarrow \mathfrak{B}(Y,X,R^{\mathrm{op}})$$

is an isomorphism of complete L-lattices for each L-context (X,Y,R). We claim that $\eta = \{\eta_{(X,Y,R)}\}$ is a natural isomorphism from $(-)^{\perp}\circ\mathfrak{B}$ to $\mathfrak{B}\circ(-)^{\perp}$. To this end, we must show that for each infomorphism $(f,g):(X,Y,R)\longrightarrow(A,B,S)$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathfrak{B}(A,B,S)^{\perp} & \xrightarrow{\ S_{\uparrow}\ } & \mathfrak{B}(B,A,S^{\mathrm{op}}) \\
\Big\downarrow{\scriptstyle \mathfrak{B}(f,g)^{\perp}} & & \Big\downarrow{\scriptstyle \mathfrak{B}(g,f)} \\
\mathfrak{B}(X,Y,R)^{\perp} & \xrightarrow[\ R_{\uparrow}\ ]{} & \mathfrak{B}(Y,X,R^{\mathrm{op}})
\end{array}$$

Since

$$\mathfrak{B}(f,g)^{\perp} = f^* : \big(S^{\downarrow}\circ S_{\uparrow}(L^A)\big)^{\mathrm{op}} \longrightarrow \big(R^{\downarrow}\circ R_{\uparrow}(L^X)\big)^{\mathrm{op}}$$

and

$$\mathfrak{B}(g,f) = g_* : S_{\uparrow}\circ S^{\downarrow}(L^B) \longrightarrow R_{\uparrow}\circ R^{\downarrow}(L^Y),$$

we only need to show that $g_*\circ S_{\uparrow} = R_{\uparrow}\circ f^*$. For all $\lambda\in S^{\downarrow}\circ S_{\uparrow}(L^A)$ and $y\in Y$,

$$\begin{aligned}
R^{\downarrow}\circ g_*\circ S_{\uparrow}(\lambda)(x) &= R^{\downarrow}\circ g^{\rightarrow}\circ S_{\uparrow}(\lambda)(x) \\
&= \bigwedge_{y\in Y}\Big(g^{\rightarrow}\circ S_{\uparrow}(\lambda)(y)\to R(x,y)\Big) \\
&= \bigwedge_{y\in Y}\bigwedge_{g(b)=y}\Big(S_{\uparrow}(\lambda)(y)\to R(x,y)\Big) \\
&= \bigwedge_{b\in B}\Big(S_{\uparrow}(\lambda)(y)\to R(x,g(b))\Big) \\
&= \bigwedge_{b\in B}\Big(S_{\uparrow}(\lambda)(y)\to S(f(x),b)\Big) \\
&= S^{\downarrow}\circ S_{\uparrow}(\lambda)(f(x)) \\
&= \lambda(f(x)) \\
&= f^*(\lambda)(x),
\end{aligned}$$

hence $R^{\downarrow}\circ g_*\circ S_{\uparrow} = f^*$, and consequently $g_*\circ S_{\uparrow} = R_{\uparrow}\circ f^*$. This completes the proof. □

6.2. **The property oriented concept lattice functor.**

Let (X,Y,R) be an L-context, define a pair of operators $(R_{\exists},R^{\forall})$ between the L-powersets of X and Y by

$$R_{\exists}: L^X\longrightarrow L^Y,\quad R_{\exists}(\mu)(y) = \bigvee_{x\in X} R(x,y)\&\mu(x)$$

and

$$R^{\forall}: L^Y\longrightarrow L^X,\quad R^{\forall}(\lambda)(x) = \bigwedge_{y\in Y}(R(x,y)\searrow\lambda(y)).$$

It is easy to check that

$$L^Y(R_{\exists}(\mu),\lambda) = L^X(\mu,R^{\forall}(\lambda))$$

for all $\mu\in L^X$ and $\lambda\in L^Y$, hence $R_{\exists}\dashv R^{\forall}: L^X\rightharpoonup L^Y$ is an L-adjunction. Therefore, $R^{\forall}\circ R_{\exists}$ is an L-closure operator on L^X.

A pair $(\mu,\lambda)\in L^X\times L^Y$ is called a *property oriented concept* if $\lambda = R_{\exists}(\mu)$ and $\mu = R^{\forall}(\lambda)$. μ is called the *object* and λ is called the *property*. The set of all property oriented concepts of (X,Y,R) is denoted by $\mathfrak{P}(X,Y,R)$. For $(\mu_1,\lambda_1),(\mu_2,\lambda_2)\in\mathfrak{P}(X,Y,R)$, let

$$\mathfrak{P}(X,Y,R)((\mu_1,\lambda_1),(\mu_2,\lambda_2)) = L^X(\mu_1,\mu_2) = L^Y(\lambda_1,\lambda_2),$$

then $\mathfrak{P}(X, Y, R)$ becomes an L-ordered set. The projection

$$\pi_1 : \mathfrak{P}(X, Y, R) \longrightarrow L^X, \quad (\mu, \lambda) \mapsto \mu$$

is clearly an L-isometry. Since the image of π_1 is exactly the set of fixed points of the L-closure operator $R^{\forall} \circ R_{\exists} : L^X \longrightarrow L^X$, we get that $\mathfrak{P}(X, Y, R)$ is isomorphic to the L-closure system $R^{\forall} \circ R_{\exists}(L^X)$ of the complete L-lattice L^X, hence $\mathfrak{P}(X, Y, R)$ is itself a complete L-lattice, called the *property oriented concept lattice* of (X, Y, R).

The projection

$$\pi_2 : \mathfrak{P}(X, Y, R) \longrightarrow L^Y, \quad (\mu, \lambda) \mapsto \lambda$$

is also an L-isometry and the image of π_2 is exactly the set of fixed points of the function $R_{\exists} \circ R^{\forall} : L^Y \longrightarrow L^Y$. Hence $\mathfrak{P}(X, Y, R)$ is also isomorphic to $R_{\exists} \circ R^{\forall}(L^Y)$[4]. Both

$$R_{\exists} : R^{\forall} \circ R_{\exists}(L^X) \longrightarrow R_{\exists} \circ R^{\forall}(L^Y)$$

and

$$R^{\forall} : R_{\exists} \circ R^{\forall}(L^Y) \longrightarrow R^{\forall} \circ R_{\exists}(L^X)$$

are isomorphisms and are inverse to each other.

In order to see the functoriality of property oriented concept lattices, we present the following proposition first.

Proposition 6.4. *Let* $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ *be an infomorphism in* L-**Ctx**. *Then* $f : (X, R^{\forall} \circ R_{\exists}) \longrightarrow (A, S^{\forall} \circ S_{\exists})$ *is continuous.*

Proof. Consider the following diagram:

$$\begin{array}{ccccc} L^X & \xrightarrow{R_{\exists}} & L^Y & \xrightarrow{R^{\forall}} & L^X \\ \downarrow{\scriptstyle f^{\rightarrow}} & & \downarrow{\scriptstyle g^{\leftarrow}} & & \downarrow{\scriptstyle f^{\rightarrow}} \\ L^A & \xrightarrow[S_{\exists}]{} & L^B & \xrightarrow[S^{\forall}]{} & L^A \end{array}$$

We must show that $f^{\rightarrow} \circ R^{\forall} \circ R_{\exists}(\mu) \leq S^{\forall} \circ S_{\exists} \circ f^{\rightarrow}(\mu)$ for all $\mu \in L^X$. The proof is similar to that of Proposition 5.1, and is omitted here. □

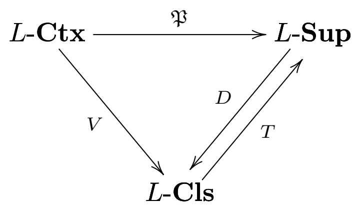
By virtue of Proposition 6.4 we obtain a functor $V : L$-**Ctx** $\longrightarrow L$-**Cls** that sends an infomorphism

$$(f, g) : (X, Y, R) \longrightarrow (A, B, S)$$

to the continuous function

$$f : (X, R^{\forall} \circ R_{\exists}) \longrightarrow (A, S^{\forall} \circ S_{\exists}).$$

If we identify $\mathfrak{P}(X, Y, R)$ with $R^{\forall} \circ R_{\exists}(L^X)$, then $\mathfrak{P}(X, Y, R) = T \circ V(X, Y, R)$. Hence $\mathfrak{P}$ is a functor from L-**Ctx** to L-**Sup**, the composition of $V : L$-**Ctx** $\longrightarrow L$-**Cls** and $T : L$-**Cls** $\longrightarrow L$-**Sup**.



The functor $\mathfrak{P} : L$-**Ctx** $\longrightarrow L$-**Sup** is called the *property oriented concept lattice functor* (or, the concept lattice functor based on rough set theory). $\mathfrak{P}$ sends an infomorphism

$$(f, g) : (X, Y, R) \longrightarrow (A, B, S)$$

to the left adjoint

$$f_* : R^{\forall} \circ R_{\exists}(L^X) \longrightarrow S^{\forall} \circ S_{\exists}(L^A)$$

[4] $R_{\exists} \circ R^{\forall} : L^Y \longrightarrow L^Y$ is an L-opening operator on L^Y and $R_{\exists} \circ R^{\forall}(L^Y)$ is an L-opening system on L^Y in the terminology of [17].

between complete L-lattices.

In the following we discuss the relationship between the concept lattice functors $\mathfrak{B}$ and $\mathfrak{P}$.

If $d \in L$ is a dualizing element, let $\neg_l, \neg_r : L \longrightarrow L$ be defined by

$$\neg_l a = d \swarrow a, \quad \neg_r a = a \searrow d.$$

For each set X, the functions $\neg_l, \neg_r$ can be extended to L^X pointwisely. In particular, the correspondence $(X, Y, R) \mapsto (X, Y, d \swarrow R)$ gives a functor $\neg_l : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Ctx}$ which leaves the morphisms untouched. The functor $\neg_r : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Ctx}$ is defined in a similar way.

It is easy to see that if d is a dualizing element, then the functors $\neg_l : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Ctx}$ and $\neg_r : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Ctx}$ are inverse to each other.

Lemma 6.5. *If d is a dualizing element, then for any L-context (X, Y, R), it holds that $R_\exists = \neg_l \circ (\neg_r R)_\uparrow$ and $R^\forall = (\neg_r R)^\downarrow \circ \neg_r$.*

Proof. Straightforward calculation by help of Proposition 2.4. □

Proposition 6.6. *If L has a dualizing element d, then $V = U \circ \neg_r$ and it has a left adjoint right inverse given by $G = \neg_l \circ F : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Ctx}$.*

Proposition 6.7. *If L has a dualizing element d, then $\mathfrak{P} = \mathfrak{B} \circ \neg_r$ and $\mathfrak{B} = \mathfrak{P} \circ \neg_l$.*

It follows immediately from the above proposition and Theorem 6.1 that if L has a dualizing element, then every complete L-lattice is of the form $\mathfrak{P}(X, Y, R)$ for some L-context (X, Y, R). The converse conclusion is also true under some mild assumptions.

Proposition 6.8. ([17] for the commutative case) *If $(L, \&)$ is a quantale with $1 \in L$ being a unit element and $0 \in L$ a cyclic element, then the following are equivalent:*

1. *0 is a dualizing element, hence L is a Girard quantale.*
2. *The functor $V : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Cls}$ has a right inverse, i.e., there exists a functor $G : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Ctx}$ such that $V \circ G = 1$.*
3. *Each complete L-lattice is isomorphic to $\mathfrak{P}(X, Y, R)$ for some L-context (X, Y, R).*

Proof. (1) ⇒ (2): Proposition 6.6.

(2) ⇒ (3): Similar to Theorem 6.1.

(3) ⇒ (1): This can be proved similarly as in the commutative case in [17] by help of L-opening operators and L-opening systems, the details are left to the reader. □

7. Conclusion

For a unital quantale $(L, \&)$, the process that sends an L-context (X, Y, R) to the formal concept lattice $\mathfrak{B}(X, Y, R)$ is a functor $\mathfrak{B} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$ from the category $L\text{-}\mathbf{Ctx}$ of L-contexts and infomorphisms to the category $L\text{-}\mathbf{Sup}$ of complete L-lattices and left adjoints. This functor can be written as the composition of a right adjoint functor $U : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Cls}$ and a left adjoint functor $T : L\text{-}\mathbf{Cls} \longrightarrow L\text{-}\mathbf{Sup}$, where $L\text{-}\mathbf{Cls}$ is the category of L-closure spaces and continuous functions.

The process that sends an L-context (X, Y, R) to the property oriented concept lattice $\mathfrak{P}(X, Y, R)$ also forms a functor $\mathfrak{P} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$.

If $(L, \&)$ has a dualizing element, then the property oriented concept lattice functor $\mathfrak{P} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$ can be written as a composition of the formal concept lattice functor $\mathfrak{B} : L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Sup}$ following a functor $L\text{-}\mathbf{Ctx} \longrightarrow L\text{-}\mathbf{Ctx}$, and vice versa.

References

[1] M. Banerjee, M. K. Chakraborty, A category for rough sets, Foundations of Computing and Decision Sciences, 18(1993) 167-180.
[2] M. Banerjee, Y. Y. Yao, A Categorial Basis for Granular Computing, Lecture Notes in Computer Science, 4482(2007) 427-434.
[3] M. Barr, ∗-Autonomous categories and linear logic, Mathematical Structures in Computer Science 1(1991) 159-178.
[4] R. Bělohlávek, Fuzzy Galois connections, Mathematical Logic Quarterly 45(1999) 497-504.

[5] R. Bělohlávek, Fuzzy closure operators, Journal of Mathematical Analysis and Applications 262(2001) 473-489.
[6] R. Bělohlávek, Concept lattices and order in fuzzy logic, Annals of Pure and Applied Logic 128(2004) 277-298.
[7] D. A. Davey, H. A. Priestley, Introduction to Lattices and Order, 2nd Edtion, Cambridge University Press, Cambridge, 2002.
[8] K. Deiters, M. Erné, Sums, products and negations of contexts and complete lattices, Algebra Universalis 60(2009) 469-496.
[9] B. Ganter, Relational Galois connections, Lecture Notes in Artificial Intelligence 4390(2007) 1-17.
[10] B. Ganter, R. Wille, Formal Concept Analysis, Mathematical Foundations, Springer, Berlin, 1999.
[11] G. Gediga, I. Düntsch, Modal-style operators in qualitative data analysis, Proceedings of the 2002 IEEE international Conference in Data Mining, 2002, pp. 155-162.
[12] G. Georgescu, A. Popescu, Concept lattices and similarity in non-commutative fuzzy logic, Fundamenta Informaticae 53(2002) 23-54.
[13] G. Georgescu, A. Popescu, Non-dual fuzzy connections, Archive for Mathematical Logic 43(2004) 1009-1039.
[14] J. Gutiérrez García, I. Mardones-Pérez, M.A. de Prada Vicente, D. Zhang, Fuzzy Galois connections categorically, Mathematical Logic Quarterly 56(2010) 131-147.
[15] R. Kent, Distributed conceptual structures, Lecture Notes in Computer Science, Vol. 2561, Springer, Berlin, 2002, pp. 104-123.
[16] M. Krötzsch, P. Hitzler, G. Q. Zhang, Morphisms in Context, Lecture Notes in Computer Science, Vol. 3596, Springer, Berlin, 2005, pp. 223-237.
[17] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, International Journal of Approximate Reasoning 50(2009) 695-707.
[18] F. W. Lawvere, Metric spaces, ganerelized logic, and closed categories, Rendiconti del Seminario Matématico e Fisico di Milano 43(1973) 135-166.
[19] S. Mac Lane, Categories for the Working Mathematician, 2nd Edition, Springer-Verlag, New York, 1998.
[20] H. Mori, Functorial properties of formal concept analysis, Lecture Notes in Computer Science 4604(2007) 505-508.
[21] A. Popescu, A general approach to fuzzy concepts, Mathematical Logic Quarterly 50(2004) 265-280.
[22] K. I. Rosenthal, Quantales and Their Applications, Pitman Research Notes in Mathematics Series, Vol. 234, Longman, Essex, 1990.
[23] I. Stubbe, Categorical structures enriched in a quantaloid: categories, distributors and functors, Theory and Applications of Categories 14(2005) 1-45.
[24] I. Stubbe, Categorical structures enriched in a quantaloid: tensored and cotensored categories, Theory and Applications of Categories 16(2006) 283-306.
[25] W. Xia, Morphismen als formale Begriffe, Darstellung und Erzeugung. Verlag Shaker, 1993.
[26] Y. Y. Yao, Concept lattices in rough set theory, in: S. Dick, L. Kurgan, W. Pedrycz, M. Reformat (Eds.), Proceedings of 2004 Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS 2004), pp. 796-801.
[27] Y. Y. Yao, A comparative study of formal concept analysis and rough set theory in data analysis, Lecture Notes in Computer Science 3066(2004) 59-68.
[28] G. Q. Zhang, Chu spaces, concept lattices, and domains, Electronic Notes in Theoretical Computer Science 83(2003) 1-17.
[29] W. N. Zhou, Generalization of L-closure spaces, Fuzzy Sets and Systems 149(2005) 415-432.

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