Introduction to Sets and Logic (MATH 1190)

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Department of Mathematics and Statistics
York University

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Outline

Course information

- Name: Introduction to Sets and Logic, MATH 1190, Section B;
- Time and Location: Thursday 7-10 pm, CLH H;
- Tutorial time and Location: Thursday 6-7 pm, CLH E;
- Moodle: `http://moodle.yorku.ca/moodle/course/view.php?id=45790` (for announcements and lecture notes)

Contact information

- Instructor: Lili Shen
- Email: shenlili@yorku.ca
- Office: N605 Ross
- Office hours: Monday and Tuesday, 9 am to 11 am. Appointments by email are also welcome.

Grading

Homework: 20%

In-class quizzes: 30%

Final exam: 50%

Grading

The following events WILL NOT affect your grades:

- "I had a big dinner, so I am late for class."
- "It's too cold and I don't want to go out for class."
- "I am tired and I have to leave early."
- Playing with your smartphone or sleeping in class.
- Making mistakes or failing to work out some exercises in your homework.

Grading

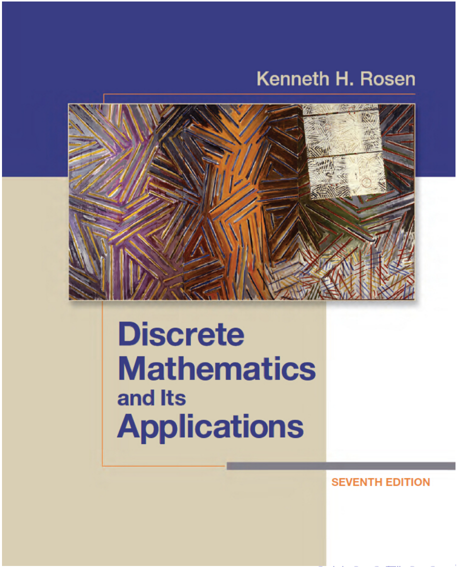
The following events WILL affect your grades:

- Making noise in class.
- Your homework is copied from a solutions guide or someone else's.
- Duplicate or extremely similar answers in a quiz or exam.

Homework

- Recommended exercises are listed at the end of each lecture note.
- The TA will collect/distribute the homework during the tutorial hour every week.
- The portion of homework in your grade only depends on whether you are serious about it. That is to say, if you hand in the homework every week, and it looks like that you have tried to work out the exercises, then you will get the 20% in your grades.

Textbook


Kenneth H. Rosen
Discrete Mathematics and Its Applications
SEVENTH EDITION

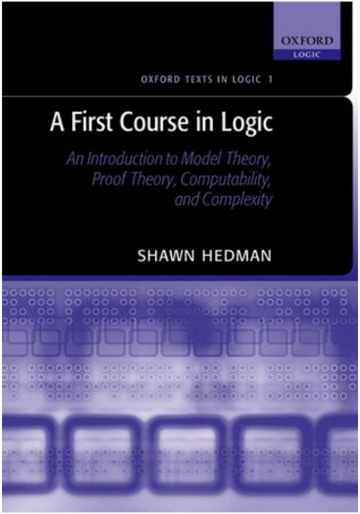
Textbook

Online resources provided by the publisher of the textbook:
`http://connect.mheducation.com/class/l-shen-fall-2014`

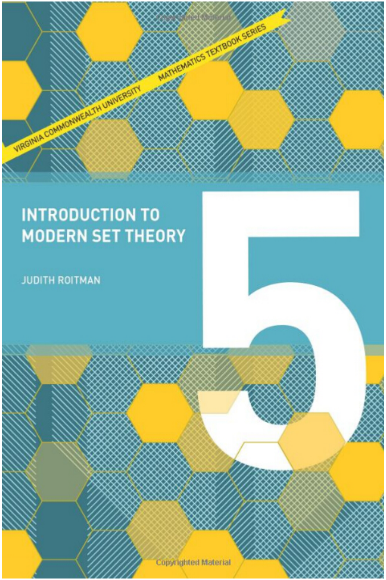
Tentative syllabus

- Chapter 1: 1.1, 1.3-1.8;
- Chapter 2: 2.1-2.5;
- Chapter 4: 4.1-4.4;
- Chapter 5: 5.1-5.3;
- Chapter 9: 9.1, 9.5, 9.6;
- Chapter 12: 12.1.

Further study



Further study



Outline of the course

For more information and possible updates about the course, please check the file

Outline of the Course

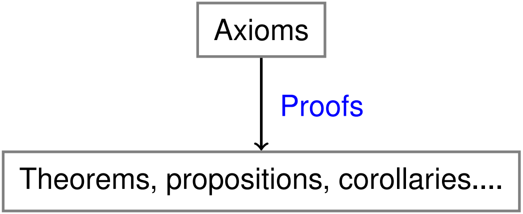
from Moodle.

Please be aware that the outline (especially the syllabus, the quiz and the exam information) will be updated from time to time according to the actual progress of the course.

Outline

Logic and proofs

Any branch of mathematics consists of the following data:



Example: Euclidean plane geometry

As an example, there are five axioms for Euclidean plane geometry:

(1) To draw a straight line from any point to any point.
(2) To produce a finite straight line continuously in a straight line.
(3) To describe a circle with any centre and distance.
(4) That all right angles are equal to one another.
(5) [The parallel postulate] That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Example: Euclidean plane geometry

The fifth axiom can be reformulated as:

(5') In a plane, through a point not on a given straight line, at most one line can be drawn that never meets the given line.

Example: Euclidean plane geometry

Starting from the five axioms, we have proved a lot of theorems, propositions or corollaries, such as:

- [Triangle Angle Sum theorem] The sum of the three angles of any triangle will always equal 180 degrees.
- [Pythagorean theorem] In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.
- ....

All these conclusions constitute the subject that you are familiar with in middle school,

"Euclidean plane geometry".

Example: non-Euclidean geometry

There are many branches of geometry in modern mathematics. The geometry that is different from the Euclidean geometry is called

"non-Euclidean geometry".

The branches of non-Euclidean geometry have one thing in common: the parallel postulate is removed from the axioms.

As a result, in non-Euclidean geometry, we do not have the theorems like

- The sum of the three angles of a triangle equals 180 degrees.

Logic and proofs

The primary goal of the "logic" part of our course is:

- Learn to read, understand and write mathematical proofs.

In other words, we will learn how to deduce theorems, propositions and corollaries from axioms or known theorems in any branch of mathematics.

Outline

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Propositions

Definition

A proposition (or statement) is a declarative sentence that is either true or false, but not both.

Propositions

The definition of a proposition contains the following points:

- A proposition is a closed sentence, which means there are no free variables in a proposition.
- Different sentences may express the same proposition (when they have the same semantics).
- A proposition can be determined true or false (but not both) within a certain context.

Examples of propositions

Example

$$1 + 1 = 2 \quad \text{and} \quad 2 + 2 = 3$$

are both propositions, while the former is true, and the latter is false.

$$x + 1 = 2 \quad \text{and} \quad x + y = z$$

are not propositions, because both of them contain free variables (called open sentences).

Examples of propositions

Example

The following sentences express an identical proposition:

- [English] Snow is white.
- [French] la neige est blanche.
- [Spanish] la nieve es blanca.
- [English] Small crystals of ice are white.

Examples of propositions

Example

The following sentence is a (false) proposition.

- Toronto is the capital of Canada.

But the following sentences are not propositions.

- What is the capital of Canada?
- Write down the capital of Canada.

Examples of propositions

Question

Is the following sentence a proposition? If so, is it true or false?

(1) The sum of the three angles of a triangle equals 180 degrees.

(2) Toronto is or is not the capital of Canada.

(3) Toronto is and is not the capital of Canada.

Examples of propositions

Solution.

(1) It is a proposition. Whether it is true or false depends on the context is Euclidean geometry or non-Euclidean geometry.

(2) It is a true proposition.

(3) It is a false proposition.

□

Examples of propositions

Question

Is the assertion

$$\text{“This statement is false”} \tag{1}$$

a proposition?

Examples of propositions

Solution.

If (1) is a proposition, then it can be assigned a truth value.

- If (1) is true, then "This statement is false" is true. Therefore (1) must be false. The hypothesis that (1) is true leads to the conclusion that (1) is false, a contradiction.
- If (1) is false, then "This statement is false" is false. Therefore (1) must be true. The hypothesis that (1) is false leads to the conclusion that (1) is true, another contradiction.

Either way, (1) is both true and false, which is a paradox. □

Examples of propositions

The statement

"This statement is false"

is known as the "liar paradox". More information can be found on Wikipedia:
http://en.wikipedia.org/wiki/Liar_paradox

Propositional logic

We usually use letters $p, q, r, s, \ldots$ to denote proposition variables (also called statement variables). This means that we write a letter p (or $q, r, s \ldots$) to denote an arbitrary proposition.

Whether a proposition is true or false is determined by its truth value. That a proposition is true is denoted by T, and that a proposition is false is denoted by F. The truth values T and F are mutually the opposite of each other.

Propositional logic

Propositional logic (or propositional calculus) is the oldest and the simplest branch of logic that deals with propositions.

In the rest of Section 1.1, we are going to learn about deriving new propositions from known propositions using logic operators (or logic connectives)

$$\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow \; \ldots.$$

These new propositions are called compound propositions.

Sometimes the terminology "compound propositions" specifically refers to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

Outline

Negation

Definition

Let p be a proposition. The negation of p, denoted by $\neg p$ (or $\overline{p}$), is the statement

"It is not the case that p."

The truth table of the negation of a proposition

The truth value of $\neg p$ is the opposite of the truth value of p.

p	$\neg p$
T	F
F	T

Examples of the negation of a proposition

Example

Find the negations of the following propositions.

(1) Michael's PC runs Linux.

(2) Vandana's smartphone has at least 32GB of memory.

(3) Vandana's smartphone has more than 32GB of memory.

(4) Every Thursday there are some students who do not come to the class MATH 1190.

Examples of the negation of a proposition

Solution.

(1) Michael's PC does not run Linux.

(2) Vandana's smartphone has less than 32GB of memory.

(3) Vandana's smartphone has at most 32GB of memory.

(4) There is at least one Thursday that all the students come to the class MATH 1190.

□

Conjunction

Definition

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition

"p and q."

Conjunction

The conjunction $p \wedge q$ is true if and only if both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples of the conjunction of two propositions

Example

"Toronto is and is not the capital of Canada"

is the conjunction of

"Toronto is the capital of Canada"

and its negation

"Toronto is not the capital of Canada".

Indeed, the conjunction of a proposition and its negation is always false.

Disjunction

Definition

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition

"p or q."

Disjunction

The disjunction $p \vee q$ is false if and only if both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples of the disjunction of two propositions

Example

"Toronto is or is not the capital of Canada"

is the disjunction of

"Toronto is the capital of Canada"

and its negation

"Toronto is not the capital of Canada".

Indeed, the disjunction of a proposition and its negation is always true.

Exclusive or

Definition

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition

"p or q, but not both."

Exclusive or

The exclusive or $p \oplus q$ is true if and only if exactly one of p and q is true.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Examples of the exclusive or of two propositions

Example

"Toronto is or is not the capital of Canada"

is the exclusive or of

"Toronto is the capital of Canada"

and its negation

"Toronto is not the capital of Canada".

Indeed, the disjunction of a proposition and its negation is always an exclusive or.

Examples of inclusive or and exclusive or

Example

Determine whether an inclusive or, or an exclusive or, is indented in the following statements.

(1) You can pay using U.S. dollars or euros.

(2) Experience with C++ or Java is required.

(3) Coffee or tea comes with dinner.

(4) When you buy a car from our company, you get \$2000 back in cash or a 2% car loan.

Examples of inclusive or and exclusive or

Solution.

Determine whether an inclusive or, or an exclusive or, is indented in the following statements.

(1) Depends on whether the bill can be paid by U.S. dollars for a portion, and euros for the remainder.
(2) Inclusive or.
(3) Exclusive or.
(4) Exclusive or.

□

Outline

Conditional statements

Definition

Let p and q be propositions. The conditional statement (or implication $p \to q$, is the proposition

$$\text{“if } p \text{ then } q\text{.”}$$

p is called the hypothesis (or antecedent, premise) and q is called the conclusion (or consequence).

Conditional statements

The implication $p \to q$ is false if and only if p is true and q is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional statements

Remark

In a conditional statement $p \to q$, there does not need to be any connection between the hypothesis p and the conclusion q. The “meaning” of $p \to q$ depends only on the truth values of p and q.

Examples of conditional statements

Example

The following conditional statements are perfectly fine, but you will not see them in ordinary English.

- If the moon is made of green cheese, then I have a million dollars.
- If $1 + 1 = 3$, then I love the course MATH 1190.

Conditional statements

Remark

A conditional statement

$$p \to q$$

is ALWAYS TRUE when p is false!

Examples of conditional statements

Example

A: If you have a million dollars, will you donate them to York University?
B: Yes! I love York University!
A: If you have twenty dollars, will you donate them to York University?
B: I am afraid not.
A: Why?
B: Because I really have twenty dollars!

Examples of conditional statements

The conditional statement

If I have a million dollars, then I donate them to York University

is always true, because the hypothesis "I have a million dollars" is false.

However, the conditional statement

If I have twenty dollars, then I donate them to York University

is true only when the conclusion "I donate them to York University" is true, because the hypothesis "I have twenty dollars" is true!

Conditional statements

There are various ways of expressing the implication $p \to q$:

- p implies q;
- q if p;
- p only if q;
- q when/whenever p;
- p is a sufficient condition for q;
- q is a necessary condition for p;
- q follows from p;
- q unless $\neg p$.

Examples of conditional statements

Example

Determine whether the conditional statements are true or false.

(1) If $2 + 2 = 4$, then $2 + 3 = 5$.

(2) If $2 + 2 = 4$, then $2 + 3 = 6$.

(3) If $2 + 2 = 3$, then $2 + 3 = 5$.

(4) If $2 + 2 = 3$, then $2 + 3 = 6$.

Examples of conditional statements

Solution.

(1) True.

(2) False.

(3) True.

(4) True.

□

Converse, contrapositive, and inverse

From a conditional statement $p \to q$ we can construct three related conditional statements:

- converse: $q \to p$;
- contrapositive: $\neg q \to \neg p$;
- inverse: $\neg p \to \neg q$.

Converse, contrapositive, and inverse

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

p	q	$\neg q \to \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p \to \neg q$
T	T	T
T	F	T
F	T	F
F	F	T

Converse, contrapositive, and inverse

Two compound propositions are called equivalent if they always have the same truth value.

- A conditional statement and its contrapositive are equivalent.
- The converse and the inverse of a conditional statement are equivalent.
- Neither the converse nor the inverse of a conditional statement is equivalent to the original conditional statement.
- Neither the converse nor the inverse of a conditional statement is equivalent to the negation of the original conditional statement.

Examples of conditional statements

Example

Consider the conditional statement

- If I have a million dollars, then I donate them to York University.

Determine its

(1) converse,

(2) contrapositive,

(3) inverse, and

(4) negation.

Examples of conditional statements

Solution.

(1) If I donate a million dollars to York University, then I have a million dollars.

(2) If I do not donate a million dollars to York University, then I do not have a million dollars.

(3) If I do not have a million dollars, then I do not donate them to York University.

(4) I have a million dollars, and I do not donate them to York University.

□

Biconditional statements

Definition

Let p and q be propositions. The biconditional statement (or bi-implications) $p \leftrightarrow q$, is the proposition

"p if and only if q."

Biconditional statements

The bi-implication $p \leftrightarrow q$ is true when p and q have the same truth values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional statements

There are various ways of expressing the bi-implication $p \leftrightarrow q$:

- p is a necessary and sufficient condition for q.
- p iff q.
- if p then q, and conversely.

It is easy to see that $p \leftrightarrow q$ is equivalent to

$$(p \to q) \wedge (q \to p).$$

Examples of biconditional statements

Example

Let p be the proposition “I have a million dollars” and q the proposition “I donate a million dollars to York University”. Then $p \leftrightarrow q$ is the statement

- I have a million dollars if and only if I donate a million dollars to York University.

This statement is true if p and q are both true or both false. It is false when p and q have opposite truth values, that is:

- you have a million dollars, but you do not donate them to York University (such as when you hate York University);
- you donate a million dollars to York University, but you do not have a million dollars (such as you are daydreaming).

Examples of the truth tables of compound propositions

Example

Construct the truth table of the compound proposition

$$(p \vee \neg q) \to (p \wedge q).$$

Examples of the truth tables of compound propositions

Solution.

p	q	$(p \vee \neg q) \to (p \wedge q)$
T	T	T
T	F	F
F	T	T
F	F	F

□

Precedence of logic operators

Now we have learned several logical operators. In order to reduce parentheses in compound propositions, we make the convention that the following precedence of logical operators is always applied when there is no parenthesis.

$$\boxed{\neg} > \boxed{\wedge, \vee} > \boxed{\rightarrow, \leftrightarrow}$$

Bits

A bit is a symbol with two possible values, 0 and 1.

A variable is called a Boolean variable if its value is either true or false. Thus, a Boolean variable can be represented by a bit, with 0 for false and 1 for true.

Bit operators

In computer science, bit operations correspond to logic connectives.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Bit strings

A bit string is a sequence of zero or more bits, such as 101010011.

The length of a bit string is the number of bits in the string.

Bit operations can be naturally extended to bit strings of the same length.

Recommended exercises

Section 1.1: 2, 4, 5, 10, 15, 18, 22, 31, 35, 37, 40.