Introduction to Sets and Logic (MATH 1190)

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Outline

Tautologies

Definition

A tautology is a compound proposition which is always true.

Examples of tautologies

Example

As we have noticed before, the disjunction of a proposition and its negation is always true, such as

“Toronto is or is not the capital of Canada.”

That is to say, $p \vee \neg p$ is a tautology.

Contradictions

Definition

A contradiction is a compound proposition which is always false.

Examples of contradictions

Example

On the contrary, the conjunction of a proposition and its negation is always false, such as

"Toronto is and is not the capital of Canada."

That is to say, $p \wedge \neg p$ is a contradiction.

Contingencies

Definition

A contingency is a proposition which is neither a tautology nor a contradiction.

The simplest examples of contingencies are p and $\neg p$.

Examples of tautologies, contradictions and contingencies

The following truth table illustrates the previous examples.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical equivalences

Definition

Two compound propositions p and q are logically equivalent, denoted by

$$p \equiv q, \quad (\text{or } p \Leftrightarrow q)$$

if $p \leftrightarrow q$ is a tautology.

Logical equivalences

Remark

(1) The symbol $\equiv$ (or $\Leftrightarrow$) is not a logic operator. $p \equiv q$ is not a compound proposition, but rather a "single" proposition

$$\text{“}p \leftrightarrow q \text{ is a tautology.”}$$

(2) Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

(3) We denote by **T** the compound proposition that is a tautology, and **F** the compound proposition that is a contradiction. For example,

$$p \vee \neg p \equiv \mathbf{T} \quad \text{and} \quad p \wedge \neg p \equiv \mathbf{F}.$$

Examples of logical equivalences

Example

Show that $p \rightarrow q \equiv \neg p \vee q$.

Proof.

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\square$

Examples of logical equivalences

Example (De Morgan laws)

(1) $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

(2) $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Proof.

We prove (1) for example.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

$\square$

Examples of logical equivalences

Example (Distributive laws)

(1) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(2) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

Examples of logical equivalences

Proof.

We prove (2) for example.

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

$\square$

Fundamental logical equivalences

- (Identity laws) $p \wedge \mathbf{T} \equiv p$, $p \vee \mathbf{F} \equiv p$.
- (Domination laws) $p \vee \mathbf{T} \equiv \mathbf{T}$, $p \wedge \mathbf{F} \equiv \mathbf{F}$.
- (Idempotent laws) $p \vee p \equiv p$, $p \wedge p \equiv p$.
- (Double negation law) $\neg\neg p \equiv p$.
- (Commutative laws) $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$.
- (Associative laws) $(p \vee q) \vee r \equiv p \vee (q \vee r)$, $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
- (Distributive laws) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
- (De Morgan laws) $\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$.
- (Absorption laws) $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$.
- (Negation laws) $p \vee \neg p \equiv \mathbf{T}$, $p \wedge \neg p \equiv \mathbf{F}$.

Fundamental logical equivalences

- $p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$.
- $p \vee q \equiv \neg p \to q$.
- $p \wedge q \equiv \neg(p \to \neg q)$.
- $\neg(p \to q) \equiv p \wedge \neg q$.
- $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.
- $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$.
- $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$.
- $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.
- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv \neg p \leftrightarrow \neg q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.
- $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

Constructing new logical equivalences

By the help these fundamental logical equivalences, we can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

To prove that $p \equiv q$, we produce a series of equivalences beginning with p and ending with q:

$$p \equiv p_1 \equiv p_2 \equiv \cdots \equiv p_n \equiv q.$$

This is a more effective way of constructing complicated logical equivalences, compared to the way of drawing truth tables.

Constructing new logical equivalences

Remark

Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

For example, the De Morgan's law says

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

Then we know that

$$\neg((p \vee \neg q) \wedge (p \vee r)) \equiv \neg(p \vee \neg q) \vee \neg(p \vee r).$$

Examples of constructing new logical equivalences

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Examples of constructing new logical equivalences

Proof.

$$
\begin{aligned}
& \neg(p \vee (\neg p \wedge q)) && \\
\equiv\ & \neg p \wedge \neg(\neg p \wedge q) && \text{(De Morgan's law)} \\
\equiv\ & \neg p \wedge (\neg\neg p \vee \neg q) && \text{(De Morgan's law)} \\
\equiv\ & \neg p \wedge (p \vee \neg q) && \text{(dobule negation law)} \\
\equiv\ & (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{(distributive law)} \\
\equiv\ & \mathbf{F} \vee (\neg p \wedge \neg q) && \text{(distributive law)} \\
\equiv\ & (\neg p \wedge \neg q) \vee \mathbf{F} && \text{(commutative law)} \\
\equiv\ & \neg p \wedge \neg q. && \text{(identity law)}
\end{aligned}
$$

$\square$

Examples of constructing new logical equivalences

Example

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Examples of constructing new logical equivalences

Proof.

$$
\begin{aligned}
& (p \wedge q) \to (p \vee q) & \\
\equiv\ & \neg(p \wedge q) \vee (p \vee q) & \text{(Table 7)} \\
\equiv\ & (\neg p \vee \neg q) \vee (p \vee q) & \text{(De Morgan's law)} \\
\equiv\ & (\neg p \vee p) \vee (\neg q \vee q) & \text{(associative and commutative law)} \\
\equiv\ & \mathbf{T} \vee \mathbf{T} & \\
\equiv\ & \mathbf{T}. & \text{(domination law)}
\end{aligned}
$$

$\square$

Propositional satisfiability

Definition

A compound proposition is satisfiable if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is unsatisfiable.

It follows immediately from the definition that

- A compound proposition is unsatisfiable if and only if its negation is a tautology.

Examples of satisfiability

Example

Both the compound propositions

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

and

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

are satisfiable, but

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

is not satisfiable.

Outline

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Propositional logic is not enough

Propositional logic is the oldest and the simplest branch of logic. However, propositional logic is not enough to deal with the following situation:

- If we have

  "All men are mortal."
  "I am a man."

  Doest it follow that "I am mortal?"

- If we have

  "All pigs are mortal."
  "I am mortal."

  Does it follow that "I am a pig?"

Predicate logic

Predicate logic uses the following new features:

- variables: $x, y, z, \ldots$;
- predicates: $P(x), Q(x), R(x, y, z), \ldots$;
- quantifiers: $\forall, \exists$.

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x, y)$;
- Variables can be replaced by elements from their domain.

Propositional functions

Remark

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain (or bound by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function P at x.

Examples of propositional functions

Example

Let $P(x)$ denote the propositional function $x > 0$ and let the domain be the integers. Then

- $P(3)$ is the proposition $3 > 0$, which is true;
- $P(0)$ is the proposition $0 > 0$, which is false;
- $P(-3)$ is the proposition $-3 > 0$, which is false.

Examples of propositional functions

Example

Let "$x + y = z$" be denoted by $R(x, y, z)$, and the domain (for all three variables) be the integers. Then:

- $R(2, 1, -5)$ is false;
- $R(3, 4, 7)$ is true;
- $R(x, 3, z)$ is not a proposition.

Examples of propositional functions

Example

Let $P(x)$ denote "$x > 0$". Then

- $P(3) \vee P(-1)$ is true;
- $P(3) \wedge P(-1)$ is false;
- $P(3) \rightarrow P(-1)$ is false;
- $P(-1) \rightarrow P(-3)$ is true;
- Neither $P(3) \wedge P(y)$ nor $P(x) \rightarrow P(y)$ is a proposition.

Outline

Quantifiers

We need quantifiers to express the meaning of English words including all and some:

- "All men are Mortal."
- "Some students skip the class."

Quantifiers

Definition

(1) The universal quantification of $P(x)$, denoted by $\forall x P(x)$, is the proposition

"$P(x)$ for all values of x in the domain."

Here $\forall$ is called the universal quantifier.

(2) The existential quantification of $P(x)$, denoted by $\exists x P(x)$, is the proposition

"There exists an element x in the domain such that $P(x)$."

Here $\exists$ is called the existential quantifier.

Quantifiers

Remark

- $\forall x P(x)$ asserts that $P(x)$ is true for every x in the domain.
- $\exists x P(x)$ asserts that $P(x)$ is true for some x in the domain.
- The quantifiers are said to bind the variable x in these expressions.
- $\exists x P(x)$ is commonly expressed in English in the following equivalent ways:
  - "There is an x such that $P(x)$."
  - "There is at least one x such that $P(x)$."
  - "For some x $P(x)$."

Quantifiers

Remark

- $P(x)$ is a propositional function (but not a proposition). It becomes a proposition when it is bound by the quantifier $\forall$ or $\exists$.
- The proposition $\forall x P(x)$ is true iff $P(x)$ is true for every x.
- The proposition $\forall x P(x)$ is false iff there is an x for which $P(x)$ is false.
- The proposition $\exists x P(x)$ is true iff there is an x for which $P(x)$ is true.
- The proposition $\exists x P(x)$ is false iff $P(x)$ is false for every x.

Examples of propositions with quantifiers

Example

Let $P(x)$ denote "$x > 0$".

- If the domain is the integers, then $\forall x P(x)$ is false, and $\exists x P(x)$ is true;
- If the domain is the positive integers, then both $\forall x P(x)$ and $\exists x P(x)$ are true.
- If the domain is the negative integers, then both $\forall x P(x)$ and $\exists x P(x)$ are false.

Uniqueness quantifiers

We write

$$\exists! x P(x)$$

for the statement that "$P(x)$ is true for one and only one x in the domain".

This is commonly expressed in English in the following equivalent ways:

- "There is a unique x such that $P(x)$."
- "There is one and only one x such that $P(x)$."

$\exists!$ is called the uniqueness quantifier.

Examples of uniqueness quantifiers

Example

Consider the integers as the domain.

- If $P(x)$ denotes "$x + 1 = 0$", then $\exists! x P(x)$ is true.
- If $P(x)$ denotes "$x > 0$", then $\exists! x P(x)$ is false (but we already know that $\exists x P(x)$ is true).

Precedence of quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators. For example,

$$\forall x P(x) \vee Q(x)$$

means

$$(\forall x P(x)) \vee Q(x)$$

rather than

$$\forall x (P(x) \vee Q(x)).$$

If you are not familiar about the precedence, please DON'T OMIT the parentheses!

Equivalences in predicate logic

Definition

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Examples of equivalences in predicate logic

Example

- $\forall x \neg\neg P(x) \equiv \forall x P(x)$.
- $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.
- $\exists ! x P(x) \equiv \exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x))$.
- If the domain consists of finite elements $x_1, x_2, \ldots, x_n$, then

$$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

$$\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

Translating from English to logic

Example

Translate the following sentence into predicate logic:

- "Every student in our class loves mathematics."

Translating from English to logic

Solution.

First we fix the domain.

- If the domain is the students in our class, define a propositional function $P(x)$ as "x loves mathematics", then the sentence is translated as

$$\forall x P(x).$$

- If the domain is all people, also define a propositional function $S(x)$ as "x is a student in our class", then the sentence is translated as

$$\forall x (S(x) \to P(x)).$$

$\square$

Translating from English to logic

Question

In the case of the domain is all people,

$$\forall x(S(x) \wedge P(x))$$

is not the correct translation. What does it mean?

Translating from English to logic

Example

Translate the following sentence into predicate logic:

- "Some students in our class love mathematics."

Translating from English to logic

Solution.

First we fix the domain.

- If the domain is the students in our class, define a propositional function $P(x)$ as "x loves mathematics", then the sentence is translated as

$$\exists x P(x).$$

- If the domain is all people, also define a propositional function $S(x)$ as "x is a student in our class", then the sentence is translated as

$$\exists x (S(x) \wedge P(x)).$$

□

Translating from English to logic

Question

In the case of the domain is all people,

$$\exists x(S(x) \to P(x))$$

is not the correct translation. What does it mean?

Recommended exercises

Section 1.3: 6, 8, 10, 26, 30, 61.

Section 1.4: 8, 10, 15, 21, 27.