Cartesian closedness of the category of real-valued sets, II

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Abstract. Let $[0,1]_*$ be the unit interval $[0,1]$ equipped with a left-continuous t-norm $*$. We prove that $[0,1]_*$-**Set** is cartesian closed if and only if $*$ is the minimum t-norm. This extends the classification for continuous t-norms established in the first part of this work to the whole left-continuous setting.

Mathematics Subject Classification. 03E72, 18D15, 18F75.

Keywords. Quantales, Quantaloids, Real-valued sets, Left-continuous t-norms, Cartesian closed categories.

1. Introduction

Building on the theory of frame-valued sets initiated by Higgs [5, 6] and Fourman–Scott [2], Höhle and his collaborators [7, 8, 9, 10, 17] developed the theory of quantale-valued sets, which associates with a unital involutive quantale Q a category

$$\mathsf{Q}\text{-}\mathbf{Set}$$

of sets valued in Q. From the perspective of enriched category theory [22], these are symmetric categories enriched in the quantaloid DQ; their morphisms are left adjoint DQ-distributors. When Q is a frame, Q-**Set** is a topos [2]. For a general quantale, however, Q-**Set** need not be a topos [12], and it can even fail to be cartesian closed [20].

In this paper, the table $[0,1]_*$ of truth-values is the unit interval equipped with a left-continuous t-norm. The first part of this work [20] established that, when $*$ is continuous, $[0,1]_*$-**Set** is cartesian closed if and only if $*$ is the minimum t-norm.

The left-continuous case cannot be treated as a formal extension of the continuous one. Indeed, Lai and Luo [15] show that, after continuity is dropped, the category $[0,1]_*$-**Cat** of categories enriched in the unital quantale

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$([0,1],*,1)$ can be cartesian closed for certain non-minimum left-continuous t-norms; see also the general quantaloid characterization of Stubbe and Yu [23]. The category $[0,1]_*$-**Cat** is a global-enrichment setting, whereas $[0,1]_*$-**Set** allows varying type data. This contrast is not explained merely by the availability of variable types: $[0,1]_*$-**Set** has symmetric $\mathsf{D}[0,1]_*$-categories with variable types as objects and left adjoint $\mathsf{D}[0,1]_*$-distributors as morphisms, whereas $[0,1]_*$-**Cat** has $[0,1]_*$-categories and $[0,1]_*$-functors. We prove that $[0,1]_*$-**Set** is cartesian closed only for the minimum t-norm; in particular, the non-minimum cases in the classification of Lai and Luo do not yield exceptions here.

Our proof differs substantially from the one in Part I [20]. Rather than using the structural description of continuous t-norms to select a local configuration and then deriving a full formula for the homs of a hypothetical exponential, we attach a canonical local obstruction to each non-idempotent $c \in [0,1]$. Namely, we consider the idempotent floor

$$a_c = \bigvee \{e \leqslant c \mid e * e = e\}.$$

The preservation of suprema by $*$ makes a_c idempotent, and hence $a_c < c$. From the pair $a_c < c$ we construct two small variable-type test objects. If a suitable exponential existed, their Cauchy completions would force two lower bounds on homs in the exponential. Evaluation gives an incompatible upper bound. Thus the argument applies throughout the left-continuous setting without a full calculation of the homs of a hypothetical exponential.

The paper is organized as follows. Section 2 recalls real-valued sets and their distributor presentation. In Section 3, we first collect the required facts about Cauchy completion, then prove the canonical obstruction and the cartesian-closedness characterization, and finally compare it with the result of Lai and Luo for $[0,1]_*$-**Cat**.

2. Real-valued sets

Throughout this paper, we consider $[0,1]_*$ as the table of truth-values, where $*$ is a left-continuous t-norm on the unit interval $[0,1]$. More generally, a binary operation $*$ on an interval $[a,b]$ is a *left-continuous t-norm* [13, 14, 1] if

- $([a,b],*,b)$ is a commutative monoid,
- $p * \bigvee\limits_{i\in I} q_i = \bigvee\limits_{i\in I} (p * q_i)$ and $(\bigvee\limits_{i\in I} p_i) * q = \bigvee\limits_{i\in I} (p_i * q)$ for all $p, q, p_i, q_i \in [a,b]$ $(i \in I)$.

We write $[a,b]_*$ for the resulting t-norm structure.

Note that for each $p \in [a,b]$, there is a Galois connection

$$(p * -) \dashv (p \to -)\colon [a,b] \longrightarrow [a,b]$$

satisfying

$$p * q \leqslant r \iff q \leqslant p \to r$$

for all $p, q, r \in [a,b]$.

Let $p \in [a,b]$. We say that p is *idempotent* if $p * p = p$.

Example 2.1. The *nilpotent minimum t-norm* $*_{\mathrm{NM}}$ on $[0,1]$ is a left-continuous but non-continuous t-norm, which is given by

$$p *_{\mathrm{NM}} q = \begin{cases} 0 & \text{if } p+q \leqslant 1, \\ p \wedge q & \text{if } p+q > 1. \end{cases}$$

Its residual is

$$p \to q = \begin{cases} 1 & \text{if } p \leqslant q, \\ (1-p) \vee q & \text{if } p > q. \end{cases}$$

In fact,

$$([a,b], *, b)$$

is a commutative unital *quantale* [18].

Given $q, u \in [a,b]$, we say that u is *divisible by* q if

$$u = q * (q \to u).$$

Proposition 2.2. *In every left-continuous t-norm* $[a,b]_*$,

$$v * (q \to u) = (q \to v) * u \tag{2.1}$$

whenever $u, v \in [a,b]$ *are divisible by* q.

Proof. Since $u = q * (q \to u)$ and $v = q * (q \to v)$, associativity and commutativity give

$$v * (q \to u) = q * (q \to v) * (q \to u) = (q \to v) * u. \qquad \blacksquare$$

Following the definition of *quantale-valued set* [7, 9, 10, 17, 16], by a $[0,1]_*$*-set* (or a $[0,1]_*$*-valued set*) we mean a set X equipped with a map

$$\alpha\colon X \times X \longrightarrow [0,1],$$

such that

(S1) $\alpha(x,y) = \alpha(x,x) * (\alpha(x,x) \to \alpha(x,y)) = \alpha(y,y) * (\alpha(y,y) \to \alpha(x,y))$,
(S2) $\alpha(x,y) = \alpha(y,x)$,
(S3) $\alpha(y,z) * (\alpha(y,y) \to \alpha(x,y)) \leqslant \alpha(x,z)$

for all $x, y, z \in X$. Here

$$\alpha(x,y)$$

is understood as the truth-value of the statement that x *is equal to* y.

In particular, it necessarily holds that

$$\alpha(x,y) \leqslant \alpha(x,x) \wedge \alpha(y,y). \tag{2.2}$$

For $x \in X$, we call the value $\alpha(x,x)$ the *type*, or *extent*, of x, which may be understood as the *extent of existence* of x.

Let (X, α) be a $[0,1]_*$-set. For $x, y \in X$, we write $x \cong y$ if

$$\alpha(x,x) = \alpha(y,y) = \alpha(x,y), \tag{2.3}$$

and we say that (X,α) is *separated* if

$$x \cong y \iff x = y.$$

Moreover, we denote by

$$X_q = \{x \in X \mid \alpha(x,x) = q\} \tag{2.4}$$

the slice of X consisting of elements of type q.

A *distributor*

$$\varphi\colon (X,\alpha) ⇸ (Y,\beta)$$

of $[0,1]_*$-sets is a function

$$\varphi\colon X \times Y \longrightarrow [0,1]$$

such that

(M1) $\varphi(x,y) = \alpha(x,x) * (\alpha(x,x) \to \varphi(x,y)) = \beta(y,y) * (\beta(y,y) \to \varphi(x,y))$,
(M2) $(\beta(y,y) \to \beta(y,y')) * \varphi(x,y) * (\alpha(x,x) \to \alpha(x',x)) \leqslant \varphi(x',y')$

for all $x,x' \in X$, $y,y' \in Y$; it necessarily holds that

$$\varphi(x,y) \leqslant \alpha(x,x) \wedge \beta(y,y). \tag{2.5}$$

The *opposite* of φ is the distributor

$$\varphi^{\circ}\colon (Y,\beta) ⇸ (X,\alpha), \quad \varphi^{\circ}(y,x) = \varphi(x,y).$$

Its composite

$$\psi \circ \varphi\colon (X,\alpha) ⇸ (Z,\gamma)$$

with another distributor $\psi\colon (Y,\beta) ⇸ (Z,\gamma)$ is given by

$$(\psi \circ \varphi)(x,z) = \bigvee_{y \in Y} \psi(y,z) * (\beta(y,y) \to \varphi(x,y)), \tag{2.6}$$

with

$$\alpha\colon (X,\alpha) ⇸ (X,\alpha) \tag{2.7}$$

serving as the identity distributor for this composition. The category of $[0,1]_*$-sets and their distributors is denoted by

$$[0,1]_*\text{-}\mathbf{Dist}.$$

$[0,1]_*$-**Dist** is a *quantaloid* [19, 21]; that is, each hom-set is a complete lattice, the order on distributors is pointwise, and the composition preserves suprema in each variable, i.e.,

$$\psi \circ \Big(\bigvee_{i \in I} \varphi_i\Big) = \bigvee_{i \in I} \psi \circ \varphi_i \quad \text{and} \quad \Big(\bigvee_{i \in I} \psi_i\Big) \circ \varphi = \bigvee_{i \in I} \psi_i \circ \varphi$$

for all distributors $\varphi, \varphi_i\colon (X,\alpha) ⇸ (Y,\beta)$, $\psi, \psi_i\colon (Y,\beta) ⇸ (Z,\gamma)$ $(i \in I)$. The corresponding right adjoints induced by the composition maps

$$(- \circ \varphi) \dashv (- \swarrow \varphi)\colon [0,1]_*\text{-}\mathbf{Dist}(X,Z) \longrightarrow [0,1]_*\text{-}\mathbf{Dist}(Y,Z)$$

and

$$(\psi \circ -) \dashv (\psi \searrow -)\colon [0,1]_*\text{-}\mathbf{Dist}(X,Z) \longrightarrow [0,1]_*\text{-}\mathbf{Dist}(X,Y)$$

satisfy

$$\psi \circ \varphi \leqslant \xi \iff \psi \leqslant \xi \swarrow \varphi \iff \varphi \leqslant \psi \searrow \xi$$

for any $\varphi\colon (X,\alpha) {\to\!\!\!\!\!\!\circ\,\,} (Y,\beta)$, $\psi\colon (Y,\beta) {\to\!\!\!\!\!\!\circ\,\,} (Z,\gamma)$, $\xi\colon (X,\alpha) {\to\!\!\!\!\!\!\circ\,\,} (Z,\gamma)$.

A *morphism*

$$\varphi\colon (X,\alpha) {\to\!\!\!\!\!\!\circ\,\,} (Y,\beta)$$

of $[0,1]_*$-sets is a left adjoint distributor, whose right adjoint is necessarily its opposite (see [3, Proposition 3.5.3] and [4, Theorem 3.7]); equivalently, it is a distributor satisfying

(M3) $\varphi \circ \varphi^\circ \leqslant \beta$,
(M4) $\alpha \leqslant \varphi^\circ \circ \varphi$.

The category of $[0,1]_*$-sets and their morphisms is denoted by

$$[0,1]_*\text{-}\mathbf{Set}.$$

We may also consider a *monotone function*

$$f\colon (X,\alpha) \longrightarrow (Y,\beta),$$

which is a map $f\colon X \longrightarrow Y$ such that

$$\alpha(x,x) = \beta(fx,fx) \quad \text{and} \quad \alpha(x,x') \leqslant \beta(fx,fx') \tag{2.8}$$

for all $x,x' \in X$. The category of $[0,1]_*$-sets and monotone functions is denoted by

$$[0,1]_*\text{-}\underline{\mathbf{Set}}.$$

Remark 2.3. Every left-continuous t-norm $[0,1]_*$ gives rise to a quantaloid $\mathsf{D}[0,1]_*$ [11, 17, 22, 16]. Its objects are the elements of $[0,1]$, and, for $p,q \in [0,1]$,

$$\mathsf{D}[0,1]_*(p,q) = \{u \in [0,1] \mid u \text{ is divisible by both } p \text{ and } q\}.$$

For $u \in \mathsf{D}[0,1]_*(p,q)$ and $v \in \mathsf{D}[0,1]_*(q,r)$, the composite is $v \circ u = v*(q \to u)$; the identity $\mathsf{D}[0,1]_*$-arrow of q is q itself, and each hom-set is ordered as a subset of $[0,1]$.

The $[0,1]_*$-sets and distributors defined above are precisely symmetric $\mathsf{D}[0,1]_*$-categories and $\mathsf{D}[0,1]_*$-distributors, respectively. Accordingly, a morphism is precisely a left adjoint $\mathsf{D}[0,1]_*$-distributor, and a monotone function is precisely a $\mathsf{D}[0,1]_*$-functor. Thus, $[0,1]_*$-**Set** is the category of symmetric $\mathsf{D}[0,1]_*$-categories and left adjoint $\mathsf{D}[0,1]_*$-distributors, while $[0,1]_*$-$\underline{\mathbf{Set}}$ is the category of symmetric $\mathsf{D}[0,1]_*$-categories and $\mathsf{D}[0,1]_*$-functors.

Every monotone function $f\colon (X,\alpha) \longrightarrow (Y,\beta)$ induces a morphism

$$f_\natural\colon (X,\alpha) {\to\!\!\!\!\!\!\circ\,\,} (Y,\beta), \quad f_\natural(x,y) = \beta(fx,y) \tag{2.9}$$

of $[0,1]_*$-sets, called the *graph* of f, whose opposite

$$f^\natural\colon (Y,\beta) {\to\!\!\!\!\!\!\circ\,\,} (X,\alpha), \quad f^\natural(y,x) = \beta(y,fx),$$

is called the *cograph* of f. Obviously, for every $[0,1]_*$-set (X,α), the identity function 1_X is monotone, and

$$\alpha = (1_X)_\natural = 1_X^\natural.$$

Hence, in order to simplify the notation, we abbreviate a $[0,1]_*$-set (X,α) to X, and write $1_X^\natural(x,y)$ instead of $\alpha(x,y)$ if no confusion arises.

For each $q\in[0,1]$, we have a one-element $[0,1]_*$-set $\{q\}$ with

$$1_{\{q\}}^\natural(q,q)=q.$$

For each distributor $\varphi\colon X \looparrowright Y$,

$$\varphi(x,-)\colon \{1_X^\natural(x,x)\} \looparrowright Y$$

and

$$\varphi(-,y)\colon X \looparrowright \{1_Y^\natural(y,y)\}$$

are distributors for all $x\in X$, $y\in Y$. In particular, $\varphi(x,y)$ can be considered as a distributor from $\{1_X^\natural(x,x)\}$ to $\{1_Y^\natural(y,y)\}$, so (M2) can be written as

(M2′) $1_Y^\natural(y,y')\circ\varphi(x,y)\circ 1_X^\natural(x',x)\leqslant\varphi(x',y')$,

(S3) can be written as

(S3′) $1_X^\natural(y,z)\circ 1_X^\natural(x,y)\leqslant 1_X^\natural(x,z)$,

and the composite distributor (2.6) can be written as

$$(\psi\circ\varphi)(x,z)=\bigvee_{y\in Y}\psi(y,z)\circ\varphi(x,y). \tag{2.10}$$

Lemma 2.4. *For $p,q\in[0,1]$, the following statements are equivalent:*

(i) *There exists a morphism $\varphi\colon\{p\}\looparrowright\{q\}$ of one-element $[0,1]_*$-sets.*

(ii) *p is divisible by q and satisfies $p=p*(q\to p)$.*

In this case, it necessarily holds that $\varphi(p,q)=p$. Therefore, a morphism between one-element $[0,1]_*$-sets, when it exists, will be denoted by

$$\overline{p}\colon\{p\}\looparrowright\{q\}.$$

When the codomain needs to be indicated, we write $\overline{p}_q$ instead of $\overline{p}$.

Proof. (i) $\Longrightarrow$ (ii): Suppose that $\varphi(p,q)=r$. Then, by (M1) and (M4) we have

$$r\leqslant p\wedge q\quad\text{and}\quad p\leqslant r*(q\to r). \tag{2.11}$$

Thus

$$p\leqslant r*(q\to r)\leqslant r\leqslant p,$$

and consequently $r=p$. By (M1),

$$p=r=q*(q\to r)=q*(q\to p),$$

so p is divisible by q. Moreover,

$$p\leqslant r*(q\to r)=p*(q\to p)\leqslant p;$$

hence

$$p=p*(q\to p). \tag{2.12}$$

(ii) $\Longrightarrow$ (i): In this case, it is easy to see that $\varphi(p,q)=p$ satisfies (M1)–(M4):

- (M1) $p=p*(p\to p)=q*(q\to p)$.
- (M2) $(q\to q)*p*(p\to p)\leqslant p$.
- (M3) $p*(p\to p)\leqslant q$.

- (M4) $p \leqslant p * (q \to p)$.

Therefore, $\varphi\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow \{q\}$ is a morphism of one-element $[0,1]_*$-sets. ■

3. Cartesian closedness of the category of real-valued sets

A *singleton* on a $[0,1]_*$-set X is a morphism $\lambda\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow X$ whose domain is a one-element $[0,1]_*$-set. A $[0,1]_*$-set is *Cauchy complete* precisely when every singleton is represented by an element.

Proposition 3.1. *(See* [21, Proposition 7.1]*.) A $[0,1]_*$-set X is Cauchy complete if and only if for each singleton $\lambda\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow X$, there exists $x \in X$ such that*

$$\lambda = 1^\natural_X(x,-).$$

In this case, necessarily $1^\natural_X(x,x) = p$.

For every $[0,1]_*$-set X, let

$$\mathsf{C}^\dagger X := \bigcup_{q\in[0,1]} [0,1]_*\text{-}\mathbf{Set}(\{q\},X)$$

be the set of all singletons on X. Its hom is

$$1^\natural_{\mathsf{C}^\dagger X}(\lambda,\lambda') = \lambda'^\circ \circ \lambda. \tag{3.1}$$

The $[0,1]_*$-set $\mathsf{C}^\dagger X$ is separated and Cauchy complete [21, Proposition 7.12]; it is called the *Cauchy completion* of X. For a singleton $\lambda\colon \{q\} \relbar\!\!\!\circ\!\!\!\rightarrow X$, one has

$$1^\natural_{\mathsf{C}^\dagger X}(\lambda,\lambda) = q. \tag{3.2}$$

The canonical monotone function

$$\mathfrak{y}^\dagger_X\colon X \longrightarrow \mathsf{C}^\dagger X, \quad \mathfrak{y}^\dagger_X x = 1^\natural_X(x,-) \tag{3.3}$$

is the unit of the Cauchy-completion reflection.

The following two elementary descriptions are obtained by the same formal calculations as in [20, Examples 3.3 and 3.4]. The difference is that Lemma 2.4 replaces the continuous-case description of morphisms between one-element $[0,1]_*$-sets:

Example 3.2. By Lemma 2.4, the Cauchy completion of the one-element $[0,1]_*$-set $\{q\}$ is

$$\mathsf{C}^\dagger\{q\} = \{\overline{p} \mid p \text{ is divisible by } q \text{ and } p = p * (q \to p)\}. \tag{3.4}$$

For $\overline{p}, \overline{p'} \in \mathsf{C}^\dagger\{q\}$,

$$1^\natural_{\mathsf{C}^\dagger\{q\}}(\overline{p},\overline{p'}) = p' * (q \to p). \tag{3.5}$$

Example 3.3. Let $X = \{x,y\}$ be a two-element $[0,1]_*$-set. Every singleton on X has a presentation

$$1^\natural_X(z,-) \circ \overline{p}\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow X,$$

where $z \in \{x,y\}$ and $\overline{p}\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow \{1^\natural_X(z,z)\}$ is a morphism. Hence

$$\mathsf{C}^\dagger X = \{1^\natural_X(z,-) \circ \overline{p} \mid z \in \{x,y\},\ \overline{p}\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow \{1^\natural_X(z,z)\}\}.$$

Let $[0,1]_*$-$\underline{\mathbf{CcSet}}$ be the full subcategory of $[0,1]_*$-$\underline{\mathbf{Set}}$ consisting of separated Cauchy complete $[0,1]_*$-sets. It is reflective in $[0,1]_*$-$\underline{\mathbf{Set}}$, with reflector $\mathsf{C}^\dagger$ (see [21, Proposition 7.14]). The following standard equivalence can be proved by the same argument as [17, Proposition 5.7(2)]; the argument remains valid for left-continuous t-norms in the present setting (cf. [21, 10]). It allows us to study the cartesian closedness of $[0,1]_*$-$\mathbf{Set}$ in $[0,1]_*$-$\underline{\mathbf{CcSet}}$.

Proposition 3.4. *The category* $[0,1]_*$-$\mathbf{Set}$ *is equivalent to* $[0,1]_*$-$\underline{\mathbf{CcSet}}$.

Recall that an object Y of a category with finite products is *exponentiable* if the functor $-\times Y$ has a right adjoint $(-)^Y$. Thus, if Z^Y exists, there is an evaluation
$$\mathrm{ev}_Z\colon Z^Y\times Y\longrightarrow Z$$
such that every monotone function $K\colon A\times Y\longrightarrow Z$ has a unique transpose
$$\mathsf{E}K\colon A\longrightarrow Z^Y$$
with $\mathrm{ev}_Z\circ(\mathsf{E}K\times 1_Y)=K$.

The category $[0,1]_*$-$\underline{\mathbf{CcSet}}$ has finite products. For $A,B\in[0,1]_*$-$\underline{\mathbf{CcSet}}$,
$$A\times B=\{(x,y)\mid x\in A,\ y\in B,\ 1_A^\natural(x,x)=1_B^\natural(y,y)\} \tag{3.6}$$
and
$$1_{A\times B}^\natural((x,y),(x',y'))=1_A^\natural(x,x')\wedge 1_B^\natural(y,y').$$

We first isolate the only consequence of the exponential adjunction that will be used below.

Lemma 3.5. *Let B be a $[0,1]_*$-set, let $Y,Z\in[0,1]_*$-$\underline{\mathbf{CcSet}}$, and suppose that Z^Y exists in $[0,1]_*$-$\underline{\mathbf{CcSet}}$. For every monotone function*
$$K\colon \mathsf{C}^\dagger B\times Y\longrightarrow Z,$$
put $k=\mathsf{E}K\colon \mathsf{C}^\dagger B\longrightarrow Z^Y$. Then
$$1_B^\natural(b,b')\leqslant 1_{Z^Y}^\natural\left(k(\mathfrak{y}_B^\dagger b),k(\mathfrak{y}_B^\dagger b')\right)$$
for all $b,b'\in B$.

Proof. The composite $k\circ\mathfrak{y}_B^\dagger\colon B\longrightarrow Z^Y$ is monotone. The assertion follows immediately from (2.8). ■

Lemma 3.6. *For $c\in[0,1]$, put*
$$a_c=\bigvee\{e\leqslant c\mid e*e=e\}.$$
Then a_c is idempotent. Consequently, if c is non-idempotent, then $a_c<c$, and every idempotent element below c is at most a_c.

Proof. Let $E_c=\{e\leqslant c\mid e*e=e\}$. Since $*$ preserves suprema in each variable,
$$a_c*a_c=\bigvee_{e,e'\in E_c}e*e'.$$
For $e,e'\in E_c$, one has $e*e'\leqslant e\leqslant a_c$, whereas the diagonal terms give $e=e*e\leqslant a_c*a_c$. Hence $a_c*a_c=a_c$. If $a_c=c$, then c is idempotent, so a

non-idempotent c satisfies $a_c < c$. The final assertion is immediate from the definition of a_c. ■

Proposition 3.7. *The category* $[0,1]_*$-$\underline{\mathbf{CcSet}}$ *is cartesian closed if and only if* $*$ *is the minimum t-norm on* $[0,1]$.

Proof. If $*$ is the minimum t-norm, then $[0,1]_*$-**Set** is the topos of $[0,1]$-valued sets over the frame $[0,1]$ [7]. It is therefore cartesian closed, and so is the equivalent category $[0,1]_*$-$\underline{\mathbf{CcSet}}$ by Proposition 3.4.

Conversely, suppose that $*$ is not the minimum t-norm. There is then a non-idempotent $c \in [0,1]$: indeed, if every element were idempotent, then $p * q = p \wedge q$ for all $p, q \in [0,1]$. By Lemma 3.6, $a_c < c$.

Let $X = \{x, x'\}$ be the $[0,1]_*$-set with

$$1_X^\natural(x,x) = 1_X^\natural(x',x') = 1, \qquad 1_X^\natural(x,x') = a_c.$$

Put $Y = \mathsf{C}^\dagger\{1\}$ and $Z = \mathsf{C}^\dagger X$. An element of Y is of the form $\overline{p}_1 \colon \{p\} \mathrel{\circ\!\!\rightarrow} \{1\}$, where p is idempotent. If $(\overline{p}_c, \overline{p}_1)$ belongs to $\mathsf{C}^\dagger\{c\} \times Y$, then p is an idempotent element below c, and hence $p \leqslant a_c$. Since a_c and p are idempotent, this also gives $a_c * p = p$.

For idempotent p, q, direct use of (3.1) gives

$$1_Z^\natural\left(1_X^\natural(x,-) \circ \overline{p}_1, 1_X^\natural(x,-) \circ \overline{q}_1\right) = p * q$$

and

$$1_Z^\natural\left(1_X^\natural(x,-) \circ \overline{p}_1, 1_X^\natural(x',-) \circ \overline{q}_1\right) = a_c * p * q.$$

Thus the following maps are monotone:

$$F \colon \mathsf{C}^\dagger\{1\} \times Y \longrightarrow Z, \qquad F(\overline{p}_1, \overline{p}_1) = 1_X^\natural(x,-) \circ \overline{p}_1,$$

$$G \colon \mathsf{C}^\dagger\{c\} \times Y \longrightarrow Z, \qquad G(\overline{p}_c, \overline{p}_1) = 1_X^\natural(x,-) \circ \overline{p}_1,$$

and

$$H \colon \mathsf{C}^\dagger\{1\} \times Y \longrightarrow Z, \qquad H(\overline{p}_1, \overline{p}_1) = 1_X^\natural(x',-) \circ \overline{p}_1.$$

For G, the preceding observation and the displayed hom calculations show monotonicity by the product formula.

Suppose that Z^Y exists. Since Z^Y is Cauchy complete, the exponential adjunction and the Cauchy-completion reflection give natural bijections (cf. [20, (4.vi)–(4.viii)])

$$Z_q^Y \cong [0,1]_*\text{-}\underline{\mathbf{CcSet}}(\mathsf{C}^\dagger\{q\} \times Y, Z)$$

for $q \in [0,1]$. Let $f \in Z_1^Y$, $g \in Z_c^Y$, and $h \in Z_1^Y$ be the elements corresponding to F, G, and H, respectively.

Let $B = \{b_0, b_1\}$ be the $[0,1]_*$-set with

$$1_B^\natural(b_0,b_0) = 1, \qquad 1_B^\natural(b_1,b_1) = c, \qquad 1_B^\natural(b_0,b_1) = c.$$

Define a map $K \colon \mathsf{C}^\dagger B \times Y \longrightarrow Z$ by

$$K\left(1_B^\natural(b_i,-) \circ \lambda, \overline{p}_1\right) = 1_X^\natural(x,-) \circ \overline{p}_1,$$

where $\lambda\colon \{p\} \relbar\!\!\!\circ\!\!\!\rightarrow \{1^\natural_B(b_i,b_i)\}$ is a morphism. By Example 3.3, this defines K on all of $\mathsf{C}^\dagger B \times Y$. If a point has two such presentations, the resulting value of K is the same for both presentations. Moreover, the hom in the source product is at most $p*q$, which is exactly the hom between the corresponding images in Z. Hence K is monotone.

For $i=0,1$, the singleton $1^\natural_B(b_i,-)$ induces a monotone function

$$j_i\colon \mathsf{C}^\dagger\{1^\natural_B(b_i,b_i)\} \longrightarrow \mathsf{C}^\dagger B, \qquad j_i(\lambda) = 1^\natural_B(b_i,-)\circ\lambda.$$

The restrictions of K along $j_0\times 1_Y$ and $j_1\times 1_Y$ are F and G, respectively. Under the natural bijections above, naturality with respect to j_0 and j_1 identifies the corresponding values of the transpose $\mathsf{E}K$ with f and g; explicitly,

$$(\mathsf{E}K)(\mathfrak{y}^\dagger_B b_0) = f, \qquad (\mathsf{E}K)(\mathfrak{y}^\dagger_B b_1) = g.$$

Lemma 3.5 therefore gives

$$c \leqslant 1^\natural_{Z^Y}(f,g).$$

The reverse inequality follows from the types of f and g, so

$$1^\natural_{Z^Y}(f,g) = c.$$

Similarly, let $B' = \{b'_0, b'_1\}$ be the $[0,1]_*$-set with

$$1^\natural_{B'}(b'_0,b'_0) = c, \qquad 1^\natural_{B'}(b'_1,b'_1) = 1, \qquad 1^\natural_{B'}(b'_0,b'_1) = c.$$

Define $K'\colon \mathsf{C}^\dagger B' \times Y \longrightarrow Z$ by

$$K'\left(1^\natural_{B'}(b'_0,-)\circ\lambda, \overline{p}_1\right) = 1^\natural_X(x,-)\circ\overline{p}_1$$

and

$$K'\left(1^\natural_{B'}(b'_1,-)\circ\lambda, \overline{p}_1\right) = 1^\natural_X(x',-)\circ\overline{p}_1.$$

If a point admits both presentations used in the definition of K', then its type is an idempotent below c, hence is at most a_c; the two prescriptions for K' therefore give the same element of Z, because $a_c * p = p$. In a mixed case, the parameter of a singleton presented through b'_0 is likewise at most a_c; consequently the mixed hom in Z is $a_c * p * q = p * q$. Since the hom in the source product is at most $p*q$, the map K' is monotone.

For $i=0,1$, define

$$j'_i\colon \mathsf{C}^\dagger\{1^\natural_{B'}(b'_i,b'_i)\} \longrightarrow \mathsf{C}^\dagger B', \qquad j'_i(\lambda) = 1^\natural_{B'}(b'_i,-)\circ\lambda.$$

The restrictions of K' along $j'_0\times 1_Y$ and $j'_1\times 1_Y$ are G and H, respectively. The same naturality argument gives

$$(\mathsf{E}K')(\mathfrak{y}^\dagger_{B'} b'_0) = g, \qquad (\mathsf{E}K')(\mathfrak{y}^\dagger_{B'} b'_1) = h.$$

Hence Lemma 3.5 gives

$$c \leqslant 1^\natural_{Z^Y}(g,h).$$

The reverse inequality follows from the types of g and h, so

$$1^\natural_{Z^Y}(g,h) = c.$$

Let $\overline{1}_1$ denote the type-1 element of Y. The monotonicity of evaluation gives

$$1^{\natural}_{Z^Y}(f,h) \leqslant 1^{\natural}_{Z}\left(\mathrm{ev}_Z(f,\overline{1}_1), \mathrm{ev}_Z(h,\overline{1}_1)\right).$$

The defining property of the transposes and the displayed calculation of the hom in Z identify the right-hand side with

$$1^{\natural}_{Z}\Big(1^{\natural}_{X}(x,-), 1^{\natural}_{X}(x',-)\Big) = a_c.$$

On the other hand, $1^{\natural}_{Z^Y}(g,g) = c$, and axiom (S3) applied to f,g,h yields

$$c * (c \to c) \leqslant 1^{\natural}_{Z^Y}(f,h).$$

This is impossible because $c * (c \to c) = c > a_c$. Hence Z^Y does not exist, and $[0,1]_*$-<u>**CcSet**</u> is not cartesian closed. ■

The main result follows at once from Propositions 3.4 and 3.7:

Theorem 3.8. *The category* $[0,1]_*$-**Set** *is cartesian closed if and only if* $*$ *is the minimum t-norm on* $[0,1]$.

Remark 3.9. Theorem 3.8 contrasts with the result of Lai and Luo [15]; see also the general quantaloid characterization of Stubbe and Yu [23]. In $[0,1]_*$-**Cat**, enrichment is global: all hom-values lie in the single quantale $[0,1]_*$, so there is no varying type data. Lai and Luo characterize the left-continuous t-norms for which $[0,1]_*$-**Cat** is cartesian closed; their classification includes t-norms different from the minimum t-norm.

By contrast, $[0,1]_*$-**Set** has symmetric $\mathsf{D}[0,1]_*$-categories with variable types as objects and left adjoint $\mathsf{D}[0,1]_*$-distributors as morphisms. The two test objects B and B' in the proof of Proposition 3.7 use the non-idempotent type c alongside type 1, a configuration unavailable in the global-enrichment setting of $[0,1]_*$-**Cat**. Thus none of the non-minimum cases in the classification of Lai and Luo yields a cartesian closed $[0,1]_*$-**Set**. This contrast is not attributable to variable types alone: symmetry and the choice of left adjoint distributors are also part of the $[0,1]_*$-**Set** framework.

Corollary 3.10. *The category* $[0,1]_*$-**Set** *is a topos if and only if* $*$ *is the minimum t-norm on* $[0,1]$.

Proof. If $*$ is the minimum t-norm, then $[0,1]_*$-**Set** is a topos [7]. Conversely, every topos is cartesian closed, so the conclusion follows from Theorem 3.8. ■

Acknowledgements

During the preparation of this manuscript, the authors used ChatGPT to assist with manuscript organization, language editing, mathematical consistency checking, and submission-readiness review. All AI-assisted suggestions were critically reviewed and, where appropriate, edited and verified by the authors, who remain fully responsible for the content of the manuscript.

Declarations

Ethical approval

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally.

Availability of data and materials

Not applicable.

Funding

The authors acknowledge support from the National Natural Science Foundation of China (Grant No. 12671561).

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