Diagonals between $\mathcal{Q}$-distributors

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Abstract

For a small quantaloid $\mathcal{Q}$, it is shown that the category of $\mathcal{Q}$-distributors and diagonals is equivalent to a quotient category of the category of $\mathcal{Q}$-interior spaces and continuous $\mathcal{Q}$-distributors. Kan adjunctions induced by $\mathcal{Q}$-distributors play a crucial role in establishing this equivalence.

Keywords: Diagonal, Quantaloid, $\mathcal{Q}$-distributor, $\mathcal{Q}$-interior space, Continuous $\mathcal{Q}$-distributor, Kan adjunction
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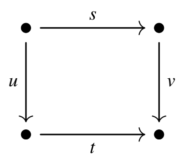
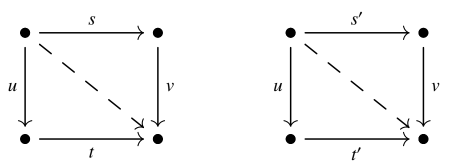
1. Introduction

Given a *(unital) quantale* [28] Q, the category $\mathbf{D}(\mathsf{Q})$ of *diagonals* [17, 27, 39] of Q has been extensively studied in the fuzzy community; see, e.g., [23, 27, 40, 16, 10, 11, 19]. More specifically, $\mathbf{D}(\mathsf{Q})$ is a *quantaloid* [30, 37, 38, 39], and categories enriched in $\mathbf{D}(\mathsf{Q})$ are precisely Q-*preordered* Q-*subsets* [10]. In particular, if Ω is a frame, then *symmetric* $\mathbf{D}(\Omega)$-categories are Ω-*sets* [6], and $\mathbf{D}(\Omega)$-categories are *skew* Ω-*sets* [4]; if $[0,\infty]$ is the *Lawvere quantale* [22], then $\mathbf{D}[0,\infty]$-categories are *(generalized) partial metric spaces* [25, 5, 17, 27, 39].

In fact, the construction of diagonals is well known in category theory. From each category $\mathcal{C}$ we may construct the *arrow category* $\mathbf{Arr}(\mathcal{C})$ [24] of $\mathcal{C}$, whose objects are $\mathcal{C}$-arrows and whose morphisms from u to v are pairs $(s : \mathrm{dom}\, u \longrightarrow \mathrm{dom}\, v,\ t : \mathrm{cod}\, u \longrightarrow \mathrm{cod}\, v)$ of $\mathcal{C}$-arrows such that the square



is commutative. There is a congruence on $\mathbf{Arr}(\mathcal{C})$, given by $(s,t) \sim (s',t')$ if the commutative squares



have the same *diagonal*; that is, if $v \circ s = t \circ u = v \circ s' = t' \circ u$. The induced quotient category

$$\mathbf{D}(\mathcal{C}) := \mathbf{Arr}(\mathcal{C})/\sim$$

is precisely the *Freyd completion* [7, 8, 9] of $\mathcal{C}$. In a nutshell, the category of diagonals of a category $\mathcal{C}$ is the Freyd completion of $\mathcal{C}$, and it has received considerable attention in the realm of category theory as well [13, 6, 42, 14, 33, 15, 41].

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Distributors [1, 2, 3] (also *profunctors* or *bimodules*) generalize functors in the same way as relations generalize maps. For a *small* quantaloid $\mathcal{Q}$, the category $\mathcal{Q}$-**Dist** of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors is again a (large) quantaloid. The aim of this paper is to investigate the category

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$$

of diagonals of the quantaloid $\mathcal{Q}$-**Dist**.

In order to explain the motivation of the results of this paper, let us recall that a *Chu transform* [34, 35] (also *infomorphism* [31, 36])

$$(f,g) : (\varphi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y) \longrightarrow (\psi : X' \mathrel{-\!\!\circ\!\!\!\rightarrow} Y')$$

between $\mathcal{Q}$-distributors is a pair of $\mathcal{Q}$-functors $f : X \longrightarrow X'$ and $g : Y' \longrightarrow Y$, such that

$$\psi \circ f_\natural = g^\natural \circ \varphi, \quad \text{or equivalently,} \quad \psi \swarrow f^\natural = g_\natural \searrow \varphi, \tag{1.i}$$

where $f_\natural$ and $f^\natural$ are the *graph* and *cograph* of f, respectively, and $\swarrow$, $\searrow$ are *left* and *right implications* in $\mathcal{Q}$-**Dist**, respectively. The two equivalent characterizations of Chu transforms in (1.i) allow us to extend the category $\mathcal{Q}$-**Chu** of $\mathcal{Q}$-distributors and Chu transforms in two directions (see [33, Proposition 3.2.1]):

$$\begin{array}{ccccc} & & \mathcal{Q}\text{-}\mathbf{Chu} & & \\ & \swarrow^{(\Box_\natural,\Box^\natural)} & & \searrow^{(\Box^\natural,\Box_\natural)} & \\ \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) & & & & \mathbf{ChuCon}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \\ \downarrow & & & & \downarrow \\ \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) & & & & \mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \end{array} \tag{1.ii}$$

In the above diagram, **ChuCon**($\mathcal{Q}$-**Dist**) and **B**($\mathcal{Q}$-**Dist**), called the categories of *Chu connections* and *back diagonals* [33] of $\mathcal{Q}$-**Dist**, dualize the constructions of **Arr**($\mathcal{Q}$-**Dist**) and **D**($\mathcal{Q}$-**Dist**), respectively. In fact, all the categories in (1.ii), expect $\mathcal{Q}$-**Chu**, are actually quantaloids. It is already known that

- **B**($\mathcal{Q}$-**Dist**) is dually equivalent to the quantaloid $\mathcal{Q}$-**Sup**(= $\mathcal{Q}$-**CCat**) of separated complete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors [33], and
- $\mathcal{Q}$-**Sup** is dually equivalent to the quantaloid $(\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}}$ of $\mathcal{Q}$-closure spaces and closed continuous $\mathcal{Q}$-distributors [32].

Hence, the combination of the main results of [32, 33] renders equivalences of quantaloids

$$\mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \simeq \mathcal{Q}\text{-}\mathbf{Sup} \simeq (\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}}^{\mathrm{op}}, \tag{1.iii}$$

which unveil the categorical and topological nature of *back diagonals between $\mathcal{Q}$-distributors*.

Since **B**($\mathcal{Q}$-**Dist**) may be considered as a dualization of **D**($\mathcal{Q}$-**Dist**) (cf. [33, Subsection 1.1]), it is natural to ask whether similar equivalences of categories could be established for **D**($\mathcal{Q}$-**Dist**). In other words, is it possible to find any categorical or topological interpretation of *diagonals between $\mathcal{Q}$-distributors*?

Unfortunately, if we take a closer look at the difference between **D**($\mathcal{Q}$-**Dist**) and **B**($\mathcal{Q}$-**Dist**), we would see that neither $\mathcal{Q}$-**Sup** nor its dual construction $\mathcal{Q}$-**Inf** (the category of separated complete $\mathcal{Q}$-categories and right adjoint $\mathcal{Q}$-functors) can be (dually) equivalent to **D**($\mathcal{Q}$-**Dist**):

(1) The canonical functor $\mathbf{ChuCon}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Sup}$ that leads to the equivalence $\mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \simeq \mathcal{Q}\text{-}\mathbf{Sup}$ (see [33, Proposition 3.3.1]) is constructed through fixed points of *Isbell adjunctions* [36], while the parallel functor $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Sup}$ (see [18, Proposition 5.1]) is constructed through fixed points of *Kan adjunctions* [36].

(2) In the special case that Q is a commutative and integral quantale, it is already known from [21] that every complete Q-category is isomorphic to the Q-category of fixed points of some Isbell adjunction, but it does not hold for Kan adjunction. Indeed, [21, Theorem 5.3] actually states that every complete Q-category is isomorphic to the Q-category of fixed points of some Kan adjunction if, and only if, Q is a *Girard quantale* [28, 43].

As a result of (1) and (2), the canonical functor from $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ to $\mathcal{Q}$-**Sup** (or $\mathcal{Q}$-**Inf**) cannot be essentially surjective on objects, and thus it cannot be an equivalence of categories. As a compromise, it is proved in [18, Theorem 5.5] that there is an equivalence of quantaloids

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})_{\rm reg}^{\rm op} \simeq (\mathcal{Q}\text{-}\mathbf{Sup})_{\rm ccd}, \tag{1.iv}$$

where $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})_{\rm reg}$ is the full subquantaloid of $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ with objects restricting to *regular* $\mathcal{Q}$-distributors, and $(\mathcal{Q}\text{-}\mathbf{Sup})_{\rm ccd}$ is the full subquantaloid of $\mathcal{Q}$-**Sup** with objects restricting to *completely distributive* $\mathcal{Q}$-categories. However, the equivalence (1.iv) cannot be extended to $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})^{\rm op}$ and $\mathcal{Q}$-**Sup**.

In spite of the difficulty in searching for the *categorical* meaning of diagonals through complete $\mathcal{Q}$-categories, in this paper we manage to present a *topological* interpretation of diagonals via $\mathcal{Q}$*-interior spaces*. As a dual notion of $\mathcal{Q}$-closure space [32], a $\mathcal{Q}$-interior space (X, a) is defined as a $\mathcal{Q}$-category X equipped with a $\mathcal{Q}$*-interior operator* [36] a on its presheaf $\mathcal{Q}$-category $\mathsf{P}X$. By constructing an adjunction

$$\mathsf{K} \dashv \mathsf{I} : \mathcal{Q}\text{-}\mathbf{Int} \longrightarrow \mathcal{Q}\text{-}\mathbf{Chu},$$

it is shown in Section 4 that the category $\mathcal{Q}$-**Int** of $\mathcal{Q}$-interior spaces and *continuous* $\mathcal{Q}$*-functors* is a retract and coreflective subcategory of $\mathcal{Q}$-**Chu** (Theorem 4.5). Explicitly, the functor K sends each $\mathcal{Q}$-distributor $\varphi : X \begin{array}{c}\circ\\[-1.2em]\longrightarrow\end{array} Y$ to the $\mathcal{Q}$-interior space $(X, \varphi^*\varphi_*)$, where

$$\varphi^* \dashv \varphi_* : \mathsf{P}X \longrightarrow \mathsf{P}Y$$

is the *Kan adjunction* [36] induced by φ.

Since the continuity of a $\mathcal{Q}$-functor between $\mathcal{Q}$-interior spaces is completely determined by its graph, it is natural to formulate the notion of *continuous* $\mathcal{Q}$*-distributor*; see Definition 5.1. In Section 5 we construct a full functor

$$\hat{\mathsf{K}} : \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist},$$

which coincides with K on objects and has a right inverse $\hat{\mathsf{I}} : \mathcal{Q}\text{-}\mathbf{IntDist} \longrightarrow \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$; hence, $\mathcal{Q}$-**IntDist** is a retract of $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$ (Proposition 5.6).

In fact, $\mathcal{Q}$-**IntDist** is a quantaloid, and there is a congruence on $\mathcal{Q}$-**IntDist**, given by $\zeta \sim \zeta' : (X, a) \begin{array}{c}\circ\\[-1.2em]\longrightarrow\end{array} (Y, b)$ if

$$\zeta^* b = \zeta'^* b,$$

which intuitively identifies "continuous maps that are indistinguishable by preimages of open sets", and we denote the quotient quantaloid by $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$. The main result of this paper, Theorem 6.4, gives an equivalence of quantaloids

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}. \tag{1.v}$$

Therefore, from the topological point of view, we may conclude that a diagonal between $\mathcal{Q}$-distributors is essentially an equivalence class of continuous $\mathcal{Q}$-distributors between $\mathcal{Q}$-interior spaces.

Moreover, we establish the discrete version of the equivalence (1.v) in Section 7 (Theorem 7.3), which is in particular applied to (classical) interior spaces and topological spaces (Corollaries 7.4 and 7.5). Finally, in Section 8 we discuss a special case, i.e., when $\mathcal{Q}$ is a Girard quantaloid. In this case, we have an isomorphism of quantaloids

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \cong \mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist})$$

by Propositions 8.2 and 8.4. Consequently, the equivalences (1.iii) and (1.v) are combined to

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq \mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \simeq (\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}} \simeq (\mathcal{Q}\text{-}\mathbf{Sup})^{\rm op}$$

if $\mathcal{Q}$ is Girard (Theorem 8.5).

2. Diagonals of a quantaloid

A *quantaloid* [30] is a category enriched in the symmetric monoidal closed category **Sup**. Explicitly, a quantaloid $\mathcal{Q}$ is a (possibly large) 2-category with its 2-cells given by order, such that each hom-set $\mathcal{Q}(p, q)$ is a complete lattice and the composition $\circ$ of $\mathcal{Q}$-arrows preserves joins on both sides, i.e.,

$$v \circ \Big(\bigvee_{i\in I} u_i\Big) = \bigvee_{i\in I} v \circ u_i, \quad \Big(\bigvee_{i\in I} v_i\Big) \circ u = \bigvee_{i\in I} v_i \circ u$$

for all $\mathcal{Q}$-arrows $u, u_i : p \longrightarrow q$ and $v, v_i : q \longrightarrow r$ $(i \in I)$. Hence, $\mathcal{Q}$ has "internal homs", denoted by $\swarrow$ and $\searrow$, as the right adjoints of the composition maps:

$$(- \circ u) \dashv (- \swarrow u) : \mathcal{Q}(p, r) \longrightarrow \mathcal{Q}(q, r) \quad \text{and} \quad (v \circ -) \dashv (v \searrow -) : \mathcal{Q}(p, r) \longrightarrow \mathcal{Q}(p, q);$$

explicitly,

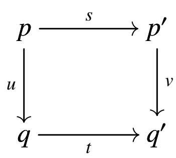
$$v \circ u \leqslant w \iff v \leqslant w \swarrow u \iff u \leqslant v \searrow w$$

for all $\mathcal{Q}$-arrows $u : p \longrightarrow q$, $v : q \longrightarrow r$, $w : p \longrightarrow r$.

A *homomorphism* of quantaloids is a functor between the underlying categories that preserves joins of arrows. A homomorphism of quantaloids is *full* (resp. *faithful*, an *equivalence* of quantaloids, an *isomorphism* of quantaloids) if the underlying functor is full (resp. faithful, an equivalence of underlying categories, an isomorphism of underlying categories).

Each quantaloid $\mathcal{Q}$ induces an arrow category $\mathbf{Arr}(\mathcal{Q})$ of $\mathcal{Q}$ with $\mathcal{Q}$-arrows as objects and pairs $(s : p \longrightarrow p',\ t : q \longrightarrow q')$ of $\mathcal{Q}$-arrows satisfying

$$v \circ s = t \circ u$$



as arrows from $u : p \longrightarrow q$ to $v : p' \longrightarrow q'$. $\mathbf{Arr}(\mathcal{Q})$ is again a quantaloid with the componentwise local order inherited from $\mathcal{Q}$.

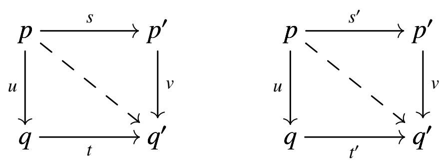
A *congruence* ϑ on a quantaloid $\mathcal{Q}$ consists of a family of equivalence relations $\vartheta_{p,q}$ on each hom-set $\mathcal{Q}(p, q)$ that is compatible with compositions and joins of $\mathcal{Q}$-arrows, i.e., $(v \circ u, v' \circ u') \in \vartheta_{p,r}$ whenever $(u, u') \in \vartheta_{p,q}$, $(v, v') \in \vartheta_{q,r}$ and $\Big(\bigvee\limits_{i\in I} u_i, \bigvee\limits_{i\in I} u'_i\Big) \in \vartheta_{p,q}$ whenever $(u_i, u'_i) \in \vartheta_{p,q}$ for all $i \in I$.

Each congruence ϑ on $\mathcal{Q}$ induces a *quotient quantaloid* $\mathcal{Q}/\vartheta$ equipped with the same objects as $\mathcal{Q}$. Compositions and joins of arrows in $\mathcal{Q}/\vartheta$ are clearly well defined, and the obvious quotient functor

$$\mathcal{Q} \longrightarrow \mathcal{Q}/\vartheta$$

is a full quantaloid homomorphism.

For arrows $(s, t), (s', t') : u \longrightarrow v$ in $\mathbf{Arr}(\mathcal{Q})$, we denote by $(s, t) \sim (s', t')$ if the commutative squares



have the same *diagonal*; that is, if

$$v \circ s = t \circ u = v \circ s' = t' \circ u.$$

"$\sim$" gives rise to a congruence on $\mathbf{Arr}(\mathcal{Q})$; the induced quotient quantaloid, denoted by

$$\mathbf{D}(\mathcal{Q}) := \mathbf{Arr}(\mathcal{Q})/\!\sim,$$

is called the quantaloid of *diagonals* [39] of $\mathcal{Q}$, which is precisely the *Freyd completion* [7, 8, 9] of $\mathcal{Q}$.

3. Quantaloid-enriched categories and their distributors

From now on, we let $\mathcal{Q}$ be a *small* quantaloid, i.e., $\mathrm{ob}\,\mathcal{Q}$ is a set. A *$\mathcal{Q}$-category* [30, 37] consists of a set X over $\mathrm{ob}\,\mathcal{Q}$, i.e., a set X equipped with a *type* map $|\text{-}| : X \longrightarrow \mathrm{ob}\,\mathcal{Q}$, and hom-arrows $\alpha(x,y) \in \mathcal{Q}(|x|,|y|)$, such that

$$1_{|x|} \leqslant \alpha(x,y) \quad \text{and} \quad \alpha(y,z) \circ \alpha(x,y) \leqslant \alpha(x,z)$$

for all $x,y,z \in X$. If (X,α) is a $\mathcal{Q}$-category, each subset $Y \subseteq X$ is equipped with the (full) *$\mathcal{Q}$-subcategory* structure inherited from α.

Each $\mathcal{Q}$-category (X,α) is endowed with an underlying (pre)order given by

$$x \leqslant y \iff |x| = |y| \text{ and } 1_{|x|} \leqslant \alpha(x,y),$$

and we write $x \cong y$ if $x \leqslant y$ and $y \leqslant x$. We say that (X,α) is *separated* (or *skeletal*) if $x = y$ whenever $x \cong y$ in X.

A *$\mathcal{Q}$-functor* (resp. *fully faithful* $\mathcal{Q}$-functor) $f : (X,\alpha) \longrightarrow (Y,\beta)$ between $\mathcal{Q}$-categories is a map $f : X \longrightarrow Y$ such that

$$|x| = |fx| \quad \text{and} \quad \alpha(x,y) \leqslant \beta(fx,fy) \quad (\text{resp. } \alpha(x,y) = \beta(fx,fy))$$

for all $x,y \in X$. With the pointwise order of $\mathcal{Q}$-functors given by

$$f \leqslant g : (X,\alpha) \longrightarrow (Y,\beta) \iff \forall x \in X : \ fx \leqslant gx,$$

$\mathcal{Q}$-categories and $\mathcal{Q}$-functors constitute a locally ordered 2-category $\mathcal{Q}$-**Cat**. A pair of $\mathcal{Q}$-functors $f : (X,\alpha) \longrightarrow (Y,\beta)$, $g : (Y,\beta) \longrightarrow (X,\alpha)$ forms an adjunction $f \dashv g$ in $\mathcal{Q}$-**Cat** if $\beta(fx,y) = \alpha(x,gy)$ for all $x \in X$, $y \in Y$.

A *$\mathcal{Q}$-distributor* $\varphi : (X,\alpha) {\circ\!\!\!\longrightarrow} (Y,\beta)$ between $\mathcal{Q}$-categories is a map that assigns to each pair $(x,y) \in X \times Y$ a $\mathcal{Q}$-arrow $\varphi(x,y) \in \mathcal{Q}(|x|,|y|)$, such that

$$\beta(y,y') \circ \varphi(x,y) \circ \alpha(x',x) \leqslant \varphi(x',y')$$

for all $x,x' \in X$, $y,y' \in Y$. With the pointwise order inherited from $\mathcal{Q}$, the locally ordered 2-category $\mathcal{Q}$-**Dist** of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors becomes a (large) quantaloid in which

$$\psi \circ \varphi : (X,\alpha) {\circ\!\!\!\longrightarrow} (Z,\gamma), \quad (\psi \circ \varphi)(x,z) = \bigvee_{y \in Y} \psi(y,z) \circ \varphi(x,y),$$
$$\xi \swarrow \varphi : (Y,\beta) {\circ\!\!\!\longrightarrow} (Z,\gamma), \quad (\xi \swarrow \varphi)(y,z) = \bigwedge_{x \in X} \xi(x,z) \swarrow \varphi(x,y),$$
$$\psi \searrow \xi : (X,\alpha) {\circ\!\!\!\longrightarrow} (Y,\beta), \quad (\psi \searrow \xi)(x,y) = \bigwedge_{z \in Z} \psi(y,z) \searrow \xi(x,z)$$

for all $\mathcal{Q}$-distributors $\varphi : (X,\alpha) {\circ\!\!\!\longrightarrow} (Y,\beta)$, $\psi : (Y,\beta) {\circ\!\!\!\longrightarrow} (Z,\gamma)$, $\xi : (X,\alpha) {\circ\!\!\!\longrightarrow} (Z,\gamma)$; the identity $\mathcal{Q}$-distributor on (X,α) is given by its hom $\alpha : (X,\alpha) {\circ\!\!\!\longrightarrow} (X,\alpha)$.

Each $\mathcal{Q}$-functor $f : (X,\alpha) \longrightarrow (Y,\beta)$ induces an adjunction $f_\natural \dashv f^\natural$ in $\mathcal{Q}$-**Dist** (i.e., $\alpha \leqslant f^\natural \circ f_\natural$ and $f_\natural \circ f^\natural \leqslant \beta$) given by

$$f_\natural : (X,\alpha) {\circ\!\!\!\longrightarrow} (Y,\beta), \quad f_\natural(x,y) = \beta(fx,y),$$
$$f^\natural : (Y,\beta) {\circ\!\!\!\longrightarrow} (X,\alpha), \quad f^\natural(y,x) = \beta(y,fx),$$

called the *graph* and *cograph* of f, respectively. Obviously, the identity $\mathcal{Q}$-distributor $\alpha : (X,\alpha) {\circ\!\!\!\longrightarrow} (X,\alpha)$ is the cograph of the identity $\mathcal{Q}$-functor $1_X : X \longrightarrow X$. Hence, in what follows

$$1_X^\natural = \alpha$$

will be our standard notation for the hom of a $\mathcal{Q}$-category $X = (X,\alpha)$ if no confusion arises. It is easy to see that

$$f \leqslant g : X \longrightarrow Y \iff g_\natural \leqslant f_\natural : X {\circ\!\!\!\longrightarrow} Y \iff f^\natural \leqslant g^\natural : Y {\circ\!\!\!\longrightarrow} X, \tag{3.i}$$

and thus both the graphs and cographs of $\mathcal{Q}$-functors are 2-functorial as

$$(-)_\natural : (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm co} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dist}, \quad (-)^\natural : (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dist},$$

where "co" refers to the dualization of 2-cells.

For each $q \in \mathrm{ob}\,\mathcal{Q}$, let $\{q\}$ denote the one-object $\mathcal{Q}$-category whose only object has type q and hom 1_q. $\mathcal{Q}$-distributors of the form $\mu : X ⇸ \{q\}$ are called *presheaves* (of type q) on X, which constitute a separated $\mathcal{Q}$-category $\mathsf{P}X$ with

$$1^\natural_{\mathsf{P}X}(\mu, \mu') = \mu' \swarrow \mu$$

for all $\mu, \mu' \in \mathsf{P}X$. The *Yoneda embedding*

$$\mathsf{y}_X : X \longrightarrow \mathsf{P}X, \ x \mapsto 1^\natural_X(-, x)$$

is clearly a fully faithful $\mathcal{Q}$-functor.

Each $\mathcal{Q}$-distributor $\varphi : X ⇸ Y$ induces a *Kan adjunction* [36]

$$\varphi^* \dashv \varphi_* : \mathsf{P}X \longrightarrow \mathsf{P}Y$$

in $\mathcal{Q}$-**Cat** with

$$\varphi^*\lambda = \lambda \circ \varphi \quad \text{and} \quad \varphi_*\mu = \mu \swarrow \varphi$$

for all $\lambda \in \mathsf{P}Y, \mu \in \mathsf{P}X$. Moreover, $(-)^* : \mathcal{Q}\text{-}\mathbf{Dist} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op}$ is 2-functorial and left adjoint to $(-)^\natural : (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Dist}$ [12], which gives rise to isomorphisms

$$\mathcal{Q}\text{-}\mathbf{Dist}(X, Y) \cong \mathcal{Q}\text{-}\mathbf{Cat}(Y, \mathsf{P}X)$$

for all $\mathcal{Q}$-categories X, Y, and we denote by

$$\widetilde{\varphi} : Y \longrightarrow \mathsf{P}X, \quad \widetilde{\varphi}y = \varphi(-, y) \tag{3.ii}$$

the transpose of each $\mathcal{Q}$-distributor $\varphi : X ⇸ Y$. It is easy to see that

$$\widetilde{\varphi} = \varphi^*\mathsf{y}_Y. \tag{3.iii}$$

In particular, each $\mathcal{Q}$-functor $f : X \longrightarrow Y$ induces an adjunction $f^\rightarrow \dashv f^\leftarrow$ in $\mathcal{Q}$-**Cat** with

$$f^\rightarrow := (f^\natural)^* : \mathsf{P}X \longrightarrow \mathsf{P}Y \quad \text{and} \quad f^\leftarrow := (f_\natural)^* = (f^\natural)_* : \mathsf{P}Y \longrightarrow \mathsf{P}X, \tag{3.iv}$$

where $(f_\natural)^* = (f^\natural)_*$ may be easily verified by routine calculation.

4. $\mathcal{Q}$-interior spaces

A $\mathcal{Q}$-*interior space* is a pair (X, a) that consists of a $\mathcal{Q}$-category X and a $\mathcal{Q}$-interior operator [36] a on $\mathsf{P}X$; that is, a $\mathcal{Q}$-functor $a : \mathsf{P}X \longrightarrow \mathsf{P}X$ with

$$a \leqslant 1_{\mathsf{P}X} \quad \text{and} \quad aa = a, \tag{4.i}$$

where we write $aa = a$ instead of $aa \cong a$ because the presheaf $\mathcal{Q}$-category $\mathsf{P}X$ is separated. We denote by

$$\mathcal{O}(X, a) := \{\mu \in \mathsf{P}X \mid a\mu = \mu\}$$

the $\mathcal{Q}$-subcategory of $\mathsf{P}X$ consisting of *open* presheaves of (X, a).

Remark 4.1. The definition of $\mathcal{Q}$-interior space here deviates from that of [20], in which a $\mathcal{Q}$-interior space is defined as a pair (X, c), with c being a $\mathcal{Q}$-*closure operator* on the *copresheaf* $\mathcal{Q}$-*category* $\mathsf{P}^\dagger X$ of X. In the case that $\mathcal{Q}$ is a *commutative quantale* [28] and X is a *discrete* $\mathcal{Q}$-category (i.e., a set), $\mathcal{Q}$-interior operators on $\mathsf{P}X$ are essentially the same as $\mathcal{Q}$-closure operators on $\mathsf{P}^\dagger X$; however, it should be noted that they may not coincide when $\mathcal{Q}$ is a general quantaloid.

A *continuous* $\mathcal{Q}$*-functor* $f : (X, a) \longrightarrow (Y, b)$ between $\mathcal{Q}$-interior spaces is a $\mathcal{Q}$-functor $f : X \longrightarrow Y$ such that

$$f^{\leftarrow} b \leqslant a f^{\leftarrow} : \mathsf{P}Y \longrightarrow \mathsf{P}X.$$

$\mathcal{Q}$-interior spaces and continuous $\mathcal{Q}$-functors constitute a 2-category $\mathcal{Q}$-**Int**, with the local order inherited from $\mathcal{Q}$-**Cat**.

The following proposition shows that continuous $\mathcal{Q}$-functors may be characterized as preimages of open presheaves staying open, and we will prove its generalized version in the next section (see Proposition 5.2):

Proposition 4.2. *Let* (X, a), (Y, b) *be* $\mathcal{Q}$*-interior spaces. For each* $\mathcal{Q}$*-functor* $f : X \longrightarrow Y$*, the following statements are equivalent:*

(i) $f : (X, a) \longrightarrow (Y, b)$ *is a continuous* $\mathcal{Q}$*-functor.*

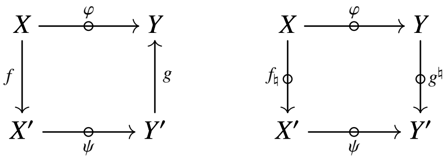
(ii) $f^{\leftarrow} b \leqslant a f^{\leftarrow} b$*, thus* $f^{\leftarrow} b = a f^{\leftarrow} b$*; that is,* $f^{\leftarrow}\lambda \in \mathcal{O}(X, a)$ *whenever* $\lambda \in \mathcal{O}(Y, b)$*.*

(iii) $b(f_\natural)_* \leqslant b(f_\natural)_* a$*, thus* $b(f_\natural)_* = b(f_\natural)_* a$*.*

Recall that a *Chu transform* [34, 35] (called *infomorphism* in [31, 36])

$$(f, g) : (\varphi : X \rightarrow\!\!\!\!\!\circ\;\, Y) \longrightarrow (\psi : X' \rightarrow\!\!\!\!\!\circ\;\, Y')$$

between $\mathcal{Q}$-distributors is a pair of $\mathcal{Q}$-functors $f : X \longrightarrow X'$ and $g : Y' \longrightarrow Y$, such that $\psi \circ f_\natural = g^\natural \circ \varphi$, or equivalently, $\psi(f-, -) = \varphi(-, g-)$.



With Chu transforms being ordered as

$$(f, g) \leqslant (f', g') : \varphi \longrightarrow \psi \iff f \leqslant f' \text{ and } g \geqslant g',$$

we obtain a locally ordered 2-category $\mathcal{Q}$-**Chu** of $\mathcal{Q}$-distributors and Chu transforms.

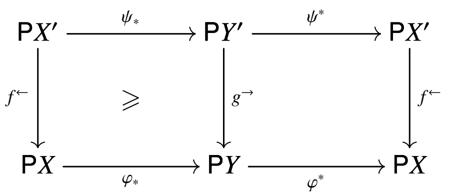
Note that the Kan adjunction $\varphi^* \dashv \varphi_* : \mathsf{P}X \longrightarrow \mathsf{P}Y$ induced by each $\mathcal{Q}$-distributor $\varphi : X \rightarrow\!\!\!\!\!\circ\;\, Y$ gives rise to a $\mathcal{Q}$-interior operator $\varphi^*\varphi_* : \mathsf{P}X \longrightarrow \mathsf{P}X$, and thus to a $\mathcal{Q}$-interior space $(X, \varphi^*\varphi_*)$. The assignment

$$(\varphi : X \rightarrow\!\!\!\!\!\circ\;\, Y) \mapsto (X, \varphi^*\varphi_*)$$

is in fact functorial from $\mathcal{Q}$-**Chu** to $\mathcal{Q}$-**Int**:

Proposition 4.3. *Let* $(f, g) : \varphi \longrightarrow \psi$ *be a Chu transform between* $\mathcal{Q}$*-distributors* $\varphi : X \rightarrow\!\!\!\!\!\circ\;\, Y$ *and* $\psi : X' \rightarrow\!\!\!\!\!\circ\;\, Y'$*. Then* $f : (X, \varphi^*\varphi_*) \longrightarrow (X', \psi^*\psi_*)$ *is a continuous* $\mathcal{Q}$*-functor.*

Proof. In order to prove $f^{\leftarrow}\psi^*\psi_* \leqslant \varphi^*\varphi_* f^{\leftarrow}$, let us consider the following diagram:



Since $f^{\rightarrow} = (f_\natural)^*$ and $g^{\rightarrow} = (g^\natural)^*$, one may check that the right square is commutative and that $g^{\rightarrow}\psi_* \leqslant \varphi_* f^{\leftarrow}$ similarly as in the proof of Proposition 5.3, by just trading ζ and η there for $f_\natural$ and $g^\natural$, respectively. The details are left to the readers. □

From Proposition 4.3 we obtain a 2-functor

$$\mathsf{K} : \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow \mathcal{Q}\text{-}\mathbf{Int}$$

that sends each Chu transform $(f,g) : (\varphi : X ⇸ Y) \longrightarrow (\psi : X' ⇸ Y')$ to the continuous $\mathcal{Q}$-functor

$$f : (X, \varphi^*\varphi_*) \longrightarrow (X', \psi^*\psi_*).$$

Conversely, from each $\mathcal{Q}$-interior space (X,a) we may construct a $\mathcal{Q}$-distributor

$$\kappa_a : X ⇸ \mathcal{O}(X,a), \quad \kappa_a(x,\mu) = \mu(x). \tag{4.ii}$$

Proposition 4.4. *A $\mathcal{Q}$-functor $f : (X,a) \longrightarrow (Y,b)$ between $\mathcal{Q}$-interior spaces is continuous if, and only if, there exists a (necessarily unique) $\mathcal{Q}$-functor $g : \mathcal{O}(Y,b) \longrightarrow \mathcal{O}(X,a)$ such that*

$$(f,g) : (\kappa_a : X ⇸ \mathcal{O}(X,a)) \longrightarrow (\kappa_b : Y ⇸ \mathcal{O}(Y,b))$$

is a Chu transform.

Proof. Suppose that $f : (X,a) \longrightarrow (Y,b)$ is continuous. From Proposition 4.2(ii) we know that the $\mathcal{Q}$-functor $f^{\leftarrow} : \mathsf{P}Y \longrightarrow \mathsf{P}X$ can be restricted to

$$f^{\leftarrow}|_{\mathcal{O}(Y,b)} : \mathcal{O}(Y,b) \longrightarrow \mathcal{O}(X,a),$$

and consequently

$$\kappa_a(x, f^{\leftarrow}|_{\mathcal{O}(Y,b)}\lambda) = (f^{\leftarrow}\lambda)(x) = \lambda(fx) = \kappa_b(fx,\lambda)$$

for all $x \in X$, $\lambda \in \mathcal{O}(Y,b)$, where the second equality holds because

$$(f^{\leftarrow}\lambda)(x) = ((f_\natural)^*\lambda)(x) = \lambda \circ f_\natural(x,-) = \lambda \circ 1_Y^\natural(fx,-) = \lambda(fx) \tag{4.iii}$$

for all $\lambda \in \mathsf{P}Y$, $x \in X$. Hence,

$$(f, f^{\leftarrow}|_{\mathcal{O}(Y,b)}) : \kappa_a \longrightarrow \kappa_b$$

is a Chu transform.

Conversely, suppose that $g : \mathcal{O}(Y,b) \longrightarrow \mathcal{O}(X,a)$ is a $\mathcal{Q}$-functor making $(f,g) : \kappa_a \longrightarrow \kappa_b$ a Chu transform. Then

$$(g\lambda)(x) = \kappa_a(x, g\lambda) = \kappa_b(fx,\lambda) = \lambda(fx) = (f^{\leftarrow}\lambda)(x)$$

for all $x \in X$, $\lambda \in \mathcal{O}(Y,b)$, where the last equality follows from (4.iii). This proves the uniqueness of g. In particular,

$$g = f^{\leftarrow}|_{\mathcal{O}(Y,b)} : \mathcal{O}(Y,b) \longrightarrow \mathcal{O}(X,a)$$

means that $f^{\leftarrow}\lambda \in \mathcal{O}(X,a)$ whenever $\lambda \in \mathcal{O}(Y,b)$. Hence, $f : (X,a) \longrightarrow (Y,b)$ is continuous by Proposition 4.2(ii). □

By Proposition 4.4, the assignment

$$(f : (X,a) \longrightarrow (Y,b)) \mapsto ((f, f^{\leftarrow}|_{\mathcal{O}(Y,b)}) : (\kappa_a : X ⇸ \mathcal{O}(X,a)) \longrightarrow (\kappa_b : Y ⇸ \mathcal{O}(Y,b)))$$

defines a fully faithful 2-functor

$$\mathsf{I} : \mathcal{Q}\text{-}\mathbf{Int} \longrightarrow \mathcal{Q}\text{-}\mathbf{Chu}$$

that embeds $\mathcal{Q}$-**Int** in $\mathcal{Q}$-**Chu** as a full 2-subcategory. In fact, this embedding is coreflective:

Theorem 4.5. *$\mathsf{K} : \mathcal{Q}\text{-}\mathbf{Chu} \longrightarrow \mathcal{Q}\text{-}\mathbf{Int}$ is a left inverse and right adjoint of $\mathsf{I} : \mathcal{Q}\text{-}\mathbf{Int} \longrightarrow \mathcal{Q}\text{-}\mathbf{Chu}$; hence, $\mathcal{Q}$-**Int** is a retract and a coreflective 2-subcategory of $\mathcal{Q}$-**Chu**.*

Proof. **Step 1.** K is a left inverse of I. For each $\mathcal{Q}$-interior space (X,a), since $\mathsf{KI}(X,a)=(X,(\kappa_a)^*(\kappa_a)_*)$, we must show that

$$a=(\kappa_a)^*(\kappa_a)_*. \tag{4.iv}$$

For each $\mu\in\mathsf{P}X$, $\lambda\in\mathcal{O}(X,a)$, we claim that

$$\mu\swarrow\lambda=a\mu\swarrow\lambda, \tag{4.v}$$

since the $\mathcal{Q}$-functoriality of a forces

$$\mu\swarrow\lambda=1^{\natural}_{\mathsf{P}X}(\lambda,\mu)\leqslant 1^{\natural}_{\mathsf{P}X}(a\lambda,a\mu)=1^{\natural}_{\mathsf{P}X}(\lambda,a\mu)=a\mu\swarrow\lambda,$$

and the reverse inequality follows from (4.i). It follows that

$$\begin{aligned}
a\mu&=\kappa_a(-,a\mu) && \text{(Equation (4.ii))}\\
&=1^{\natural}_{\mathcal{O}(X,a)}(-,a\mu)\circ\kappa_a\\
&=\bigvee_{\lambda\in\mathcal{O}(X,a)}1^{\natural}_{\mathcal{O}(X,a)}(\lambda,a\mu)\circ\kappa_a(-,\lambda)\\
&=\bigvee_{\lambda\in\mathcal{O}(X,a)}(a\mu\swarrow\lambda)\circ\kappa_a(-,\lambda)\\
&=\bigvee_{\lambda\in\mathcal{O}(X,a)}(\mu\swarrow\lambda)\circ\kappa_a(-,\lambda) && \text{(Equation (4.v))}\\
&=\bigvee_{\lambda\in\mathcal{O}(X,a)}(\mu\swarrow\kappa_a(-,\lambda))\circ\kappa_a(-,\lambda) && \text{(Equation (4.ii))}\\
&=(\mu\swarrow\kappa_a)\circ\kappa_a\\
&=(\kappa_a)^*(\kappa_a)_*\mu
\end{aligned}$$

for all $\mu\in\mathsf{P}X$, as desired.

Step 2. K is a right adjoint of I. For each $\mathcal{Q}$-interior space (X,a), since $\mathsf{KI}(X,a)=(X,a)$, it suffices to show that the identity natural transformation

$$\{1_X:(X,a)\longrightarrow\mathsf{KI}(X,a)\mid(X,a)\in\mathrm{ob}(\mathcal{Q}\text{-}\mathbf{Int})\}$$

is the unit of the adjunction $\mathsf{I}\dashv\mathsf{K}$; that is, for each $\mathcal{Q}$-distributor $\psi:Y\xrightarrow{\circ}Z$ and continuous $\mathcal{Q}$-functor

$$h:(X,a)\longrightarrow\mathsf{K}(\psi:Y\xrightarrow{\circ}Z)=(Y,\psi^*\psi_*),$$

there exists a unique Chu transform

$$(f,g):\mathsf{I}(X,a)=(\kappa_a:X\xrightarrow{\circ}\mathcal{O}(X,a))\longrightarrow(\psi:Y\xrightarrow{\circ}Z)$$

such that the triangle

$$\begin{array}{ccc}
(X,a) & \xrightarrow{\ 1_X\ } & \mathsf{KI}(X,a)\\
 & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \mathsf{K}(f,g)}\\
 & & (Y,\psi^*\psi_*)
\end{array}$$

is commutative. Since $\mathsf{K}(f,g)=f$, it remains to show that there exists a unique $\mathcal{Q}$-functor $g:Z\longrightarrow\mathcal{O}(X,a)$ such that

$$(h,g):(\kappa_a:X\xrightarrow{\circ}\mathcal{O}(X,a))\longrightarrow(\psi:Y\xrightarrow{\circ}Z)$$

is a Chu transform. To this end, we define

$$g:=h^{\leftarrow}\widetilde{\psi}:Z\longrightarrow\mathsf{P}Y\longrightarrow\mathcal{O}(X,a).$$

First, g is well defined. Indeed, for each $z \in Z$, by Equation (3.iii) we have

$$\widetilde{\psi}z = \psi^*\mathsf{y}_Z z = \psi^*\psi_*\psi^*\mathsf{y}_Z z \in \mathcal{O}(Y, \psi^*\psi_*),$$

and together with the continuity of h we deduce that $gz = h^{\leftarrow}\widetilde{\psi}z \in \mathcal{O}(X, a)$.

Second, (h, g) is a Chu transform. Indeed,

$$\kappa_a(x, gz) = (gz)(x) = (h^{\leftarrow}\widetilde{\psi}z)(x) = (\widetilde{\psi}z)(hx) = \psi(hx, z)$$

for all $x \in X$, $z \in Z$, where the penultimate equality follows from (4.iii).

Finally, for the uniqueness of g, suppose that $g' : Z \longrightarrow \mathcal{O}(X, a)$ is another $\mathcal{Q}$-functor making $(h, g') : \kappa_a \longrightarrow \psi$ a Chu transform. Then

$$(g'z)(x) = \kappa_a(x, g'z) = \psi(hx, z) = (\widetilde{\psi}z)(hx) = (h^{\leftarrow}\widetilde{\psi}z)(x) = (gz)(x)$$

for all $z \in Z$, $x \in X$. Hence $g' = g$. □

Remark 4.6. As pointed out by the anonymous referee, $\mathcal{Q}$-**Chu** may be identified with the *comma category* (cf. [24, Section II.6])

$$\mathcal{Q}\text{-}\mathbf{Cat} \downarrow (-)^{\leftarrow},$$

where $(-)^{\leftarrow} : (\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ is the functor sending each $\mathcal{Q}$-functor $f : X \longrightarrow Y$ to $f^{\leftarrow} : \mathsf{P}Y \longrightarrow \mathsf{P}X$ (cf. (3.iv)). Under this identification, $\mathsf{I} : \mathcal{Q}\text{-}\mathbf{Int} \longrightarrow \mathcal{Q}\text{-}\mathbf{Chu}$ actually sends each $\mathcal{Q}$-interior space (X, a) to the inclusion $\mathcal{Q}$-functor $\mathcal{O}(X, a) \hookrightarrow \mathsf{P}X$; that is, the transpose $\widetilde{\kappa_a}$ of the $\mathcal{Q}$-distributor $\kappa_a : X ⇸ \mathcal{O}(X, a)$.

Moreover, it is stated in [26] that a topological space is an *extensional Chu space* whose columns are closed under arbitrary union and finite intersection. This observation may be generalized to our context as follows: a $\mathcal{Q}$-distributor $\varphi : X ⇸ Y$ can be identified with a $\mathcal{Q}$-interior space if its transpose $\widetilde{\varphi} : Y \longrightarrow \mathsf{P}X$ is a fully faithful $\mathcal{Q}$-functor, and if the image

$$\mathsf{Im}\,\widetilde{\varphi} := \{\widetilde{\varphi}y \mid y \in Y\}$$

of $\widetilde{\varphi}$, as a $\mathcal{Q}$-subcategory of $\mathsf{P}X$, is closed under suprema in $\mathsf{P}X$ (cf. [31, Proposition 4.1.8]); where the latter requirement may also be equivalently expressed as

$$\sup\nolimits_{\mathsf{P}X}\widetilde{\varphi}^{\rightarrow}\lambda \in \mathsf{Im}\,\widetilde{\varphi}$$

for all $\lambda \in \mathsf{P}Y$. In this case, it is not difficult to prove that

$$\mathcal{O}(X, \varphi^*\varphi_*) = \mathsf{Im}\,\widetilde{\varphi},$$

so that the $\mathcal{Q}$-interior space identified with $\varphi : X ⇸ Y$ is precisely $(X, \varphi^*\varphi_*)$.

5. Continuous $\mathcal{Q}$-distributors

Since $f^{\leftarrow} = (f_\natural)^*$, the continuity of a $\mathcal{Q}$-functor $f : (X, a) \longrightarrow (Y, b)$ between $\mathcal{Q}$-interior spaces is completely determined by its graph $f_\natural : X ⇸ Y$, i.e.,

$$(f_\natural)^* b \leqslant a(f_\natural)^* : \mathsf{P}Y \longrightarrow \mathsf{P}X.$$

If $f_\natural$ is replaced by an arbitrary $\mathcal{Q}$-distributor $\zeta : X ⇸ Y$, we come to the following definition:

Definition 5.1. A *continuous $\mathcal{Q}$-distributor* $\zeta : (X, a) ⇸ (Y, b)$ between $\mathcal{Q}$-interior spaces is a $\mathcal{Q}$-distributor $\zeta : X ⇸ Y$ such that

$$\zeta^* b \leqslant a\zeta^* : \mathsf{P}Y \longrightarrow \mathsf{P}X.$$

With the local order inherited from $\mathcal{Q}$-**Dist**, $\mathcal{Q}$-interior spaces and continuous $\mathcal{Q}$-distributors constitute a (large) quantaloid $\mathcal{Q}$-**IntDist**, for it is easy to verify that compositions and joins of continuous $\mathcal{Q}$-distributors are still continuous. There is clearly a 2-functor

$$(-)_\natural : (\mathcal{Q}\text{-}\mathbf{Int})^{\mathrm{co}} \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist}$$

sending each continuous $\mathcal{Q}$-functor $f : (X, a) \longrightarrow (Y, b)$ to the continuous $\mathcal{Q}$-distributor $f_\natural : (X, a) ⇸ (Y, b)$.

Proposition 5.2. *Let (X, a), (Y, b) be $\mathcal{Q}$-interior spaces. For each $\mathcal{Q}$-distributor $\zeta : X \nrightarrow Y$, the following statements are equivalent:*

(i) *$\zeta : (X, a) \nrightarrow (Y, b)$ is a continuous $\mathcal{Q}$-distributor.*

(ii) *$\zeta^* b \leqslant a\zeta^* b$, thus $\zeta^* b = a\zeta^* b$; that is, $\zeta^*\lambda \in \mathcal{O}(X, a)$ whenever $\lambda \in \mathcal{O}(Y, b)$.*

(iii) *$b\zeta_* \leqslant b\zeta_* a$, thus $b\zeta_* = b\zeta_* a$.*

Proof. (i) $\Longrightarrow$ (ii): If $\zeta^* b \leqslant a\zeta^*$, then $\zeta^* b = \zeta^* bb \leqslant a\zeta^* b$.

(ii) $\Longrightarrow$ (iii): This follows from $b = bb \leqslant b\zeta_*\zeta^* b \leqslant b\zeta_* a\zeta^* b \leqslant b\zeta_* a\zeta^*$ and $\zeta^* \dashv \zeta_*$.

(iii) $\Longrightarrow$ (i): $\zeta^* b \leqslant a\zeta^*$ follows immediately from $b \leqslant b\zeta_*\zeta^* \leqslant b\zeta_* a\zeta^* \leqslant \zeta_* a\zeta^*$ and $\zeta^* \dashv \zeta_*$. □

As a generalized version of Proposition 4.3 we have:

Proposition 5.3. *Each commutative square*

$$\begin{array}{ccc} X & \overset{\zeta}{\nrightarrow} & X' \\ {\scriptstyle\varphi}\downarrow\!\!\!\circ & & \circ\!\!\!\downarrow{\scriptstyle\psi} \\ Y & \underset{\eta}{\nrightarrow} & Y' \end{array}$$

*in $\mathcal{Q}$-**Dist** induces a continuous $\mathcal{Q}$-distributor $\zeta : (X, \varphi^*\varphi_*) \nrightarrow (X', \psi^*\psi_*)$.*

Proof. In order to prove $\zeta^*\psi^*\psi_* \leqslant \varphi^*\varphi_*\zeta^*$, let us consider the following diagram:

$$\begin{array}{ccccc} \mathsf{P}X' & \xrightarrow{\psi_*} & \mathsf{P}Y' & \xrightarrow{\psi^*} & \mathsf{P}X' \\ {\scriptstyle\zeta^*}\downarrow & \geqslant & \downarrow{\scriptstyle\eta^*} & & \downarrow{\scriptstyle\zeta^*} \\ \mathsf{P}X & \xrightarrow[\varphi_*]{} & \mathsf{P}Y & \xrightarrow[\varphi^*]{} & \mathsf{P}X \end{array}$$

The commutativity of the right square follows immediately from the functoriality of $(-)^* : (\mathcal{Q}\text{-}\mathbf{Dist})^{\mathrm{op}} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$, and it remains to verify $\eta^*\psi_* \leqslant \varphi_*\zeta^*$. Indeed,

$$\begin{aligned} \eta^*\psi_*\mu' &= (\mu' \swarrow \psi) \circ \eta \\ &\leqslant ((\mu' \swarrow \psi) \circ \eta \circ \varphi) \swarrow \varphi \\ &= ((\mu' \swarrow \psi) \circ \psi \circ \zeta) \swarrow \varphi \\ &\leqslant (\mu' \circ \zeta) \swarrow \varphi \\ &= \varphi_*\zeta^*\mu' \end{aligned}$$

for all $\mu' \in \mathsf{P}X'$, and thus the conclusion follows. □

Proposition 5.3 actually gives rise to a quantaloid homomorphism

$$\hat{\mathsf{K}} : \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist}$$

that sends each arrow $(\zeta, \eta) : (\varphi : X \nrightarrow Y) \longrightarrow (\psi : X' \nrightarrow Y')$ in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$ to the continuous $\mathcal{Q}$-distributor $\zeta : (X, \varphi^*\varphi_*) \nrightarrow (X', \psi^*\psi_*)$.

Since every Chu transform $(f, g) : \varphi \longrightarrow \psi$ induces an arrow $(f_\natural, g^\natural) : \varphi \longrightarrow \psi$ in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$, there is a 2-functor

$$(\square_\natural, \square^\natural) : (\mathcal{Q}\text{-}\mathbf{Chu})^{\mathrm{co}} \longrightarrow \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}), \quad (f, g) \mapsto (f_\natural, g^\natural)$$

which is neutral on objects. As the commutative square

$$\begin{array}{ccc} \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) & \xrightarrow{\ \hat{\mathsf{K}}\ } & \mathcal{Q}\text{-}\mathbf{IntDist} \\ \big\uparrow{\scriptstyle (\Box_\natural,\Box^\natural)} & & \big\uparrow{\scriptstyle (-)_\natural} \\ (\mathcal{Q}\text{-}\mathbf{Chu})^{\mathrm{co}} & \xrightarrow[\ \mathsf{K}^{\mathrm{co}}\ ]{} & (\mathcal{Q}\text{-}\mathbf{Int})^{\mathrm{co}} \end{array} \tag{5.i}$$

reveals, $\hat{\mathsf{K}}$ may be viewed as an extension of the functor K. Moreover:

Proposition 5.4. $\hat{\mathsf{K}} : \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist}$ *is a full quantaloid homomorphism.*

Proof. It remains to show that $\hat{\mathsf{K}}$ is full. Given $\mathcal{Q}$-distributors $\varphi : X \to\!\!\!\!\!\circ\ Y$, $\psi : X' \to\!\!\!\!\!\circ\ Y'$, we need to show that

$$\hat{\mathsf{K}} : \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})(\varphi,\psi) \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist}((X,\varphi^*\varphi_*),(X',\psi^*\psi_*))$$

is surjective. To this end, for any continuous $\mathcal{Q}$-distributor $\zeta : (X,\varphi^*\varphi_*) \to\!\!\!\!\!\circ\ (X',\psi^*\psi_*)$, we must find a $\mathcal{Q}$-distributor $\eta : Y \to\!\!\!\!\!\circ\ Y'$ such that $(\zeta,\eta) : \varphi \longrightarrow \psi$ is an arrow in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$. Indeed, let

$$\eta := (\psi \circ \zeta) \swarrow \varphi : Y \to\!\!\!\!\!\circ\ Y'.$$

Then

$$\begin{aligned} \eta(-,y') \circ \varphi &= \varphi^*\varphi_*\zeta^*\psi(-,y') && (\eta = (\psi\circ\zeta)\swarrow\varphi) \\ &= \varphi^*\varphi_*\zeta^*\psi^*\mathsf{y}_{Y'}y' && \text{(Equations (3.ii) and (3.iii))} \\ &= \varphi^*\varphi_*\zeta^*\psi^*\psi_*\psi^*\mathsf{y}_{Y'}y' && (\psi^* \dashv \psi_*) \\ &= \zeta^*\psi^*\psi_*\psi^*\mathsf{y}_{Y'}y' && \text{(Proposition 5.2(ii))} \\ &= \zeta^*\psi^*\mathsf{y}_{Y'}y' && (\psi^* \dashv \psi_*) \\ &= \psi(-,y') \circ \zeta && \text{(Equations (3.ii) and (3.iii))} \end{aligned}$$

for all $y' \in Y'$, as desired. □

Analogously to (5.i), there is a quantaloid homomorphism

$$\hat{\mathsf{I}} : \mathcal{Q}\text{-}\mathbf{IntDist} \longrightarrow \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$$

such that the square

$$\begin{array}{ccc} \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) & \xleftarrow{\ \hat{\mathsf{I}}\ } & \mathcal{Q}\text{-}\mathbf{IntDist} \\ \big\uparrow{\scriptstyle (\Box_\natural,\Box^\natural)} & & \big\uparrow{\scriptstyle (-)_\natural} \\ (\mathcal{Q}\text{-}\mathbf{Chu})^{\mathrm{co}} & \xleftarrow[\ \mathsf{I}^{\mathrm{co}}\ ]{} & (\mathcal{Q}\text{-}\mathbf{Int})^{\mathrm{co}} \end{array} \tag{5.ii}$$

is commutative, and thus extends $\mathsf{I} : \mathcal{Q}\text{-}\mathbf{Int} \longrightarrow \mathcal{Q}\text{-}\mathbf{Chu}$:

Proposition 5.5. *For each continuous $\mathcal{Q}$-distributor $\zeta : (X,a) \to\!\!\!\!\!\circ\ (Y,b)$ between $\mathcal{Q}$-interior spaces,*

$$(\zeta,(\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}) : (\kappa_a : X \to\!\!\!\!\!\circ\ \mathcal{O}(X,a)) \longrightarrow (\kappa_b : Y \to\!\!\!\!\!\circ\ \mathcal{O}(Y,b)))$$

is an arrow in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$*, where $(\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}$ is the restriction of $(\zeta^*)^\natural : \mathsf{P}X \to\!\!\!\!\!\circ\ \mathsf{P}Y$ on $\mathcal{O}(X,a)$ and $\mathcal{O}(Y,b)$.*

Proof. Note that the $\mathcal{Q}$-distributor $(\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)} : \mathcal{O}(X,a) \to\!\!\!\!\!\circ\ \mathcal{O}(Y,b)$ is well defined since $\zeta^*\lambda \in \mathcal{O}(X,a)$ whenever $\lambda \in \mathcal{O}(Y,b)$ by Proposition 5.2(ii). The conclusion then follows from

$$\kappa_b(-,\lambda) \circ \zeta = \lambda \circ \zeta = \zeta^*\lambda = \kappa_a(-,\zeta^*\lambda) = 1^\natural_{\mathcal{O}(X,a)}(-,\zeta^*\lambda) \circ \kappa_a = (\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}(-,\lambda) \circ \kappa_a$$

for all $\lambda \in \mathcal{O}(Y,b)$. □

The following proposition is an immediate consequence of Theorem 4.5 in combination with the definitions of $\hat{\mathsf{K}}$ and $\hat{\mathsf{I}}$:

Proposition 5.6. *$\hat{\mathsf{K}} : \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow \mathcal{Q}\text{-}\mathbf{IntDist}$ is a left inverse of $\hat{\mathsf{I}} : \mathcal{Q}\text{-}\mathbf{IntDist} \longrightarrow \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$; hence, $\mathcal{Q}\text{-}\mathbf{IntDist}$ is a retract of $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$.*

Remark 5.7. As pointed out by the anonymous referee, it is worth considering the comma category

$$\mathcal{Q}\text{-}\mathbf{Cat} \downarrow (-)^*$$

here, where $(-)^* : (\mathcal{Q}\text{-}\mathbf{Dist})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ sends each $\mathcal{Q}$-distributor $\varphi : X \,{-\!\!\circ\!\!\!\rightarrow}\, Y$ to the $\mathcal{Q}$-functor $\varphi^* : \mathsf{P}Y \longrightarrow \mathsf{P}X$. This comma category may be identified with the category having $\mathcal{Q}$-distributors as objects and pairs $(\zeta : X \,{-\!\!\circ\!\!\!\rightarrow}\, X',\ g : Y' \longrightarrow Y)$ consisting of a $\mathcal{Q}$-distributor and a $\mathcal{Q}$-functor satisfying

$$\psi \circ \zeta = g^{\natural} \circ \varphi$$

as morphisms from $\varphi : X \,{-\!\!\circ\!\!\!\rightarrow}\, Y$ to $\psi : X' \,{-\!\!\circ\!\!\!\rightarrow}\, Y'$. From Proposition 5.5 it is easy to see that $\mathcal{Q}$-**IntDist** can be embedded into $\mathcal{Q}\text{-}\mathbf{Cat} \downarrow (-)^*$ by sending each continuous $\mathcal{Q}$-distributor $\zeta : (X,a) \,{-\!\!\circ\!\!\!\rightarrow}\, (Y,b)$ to $(\zeta, \zeta^*|_{\mathcal{O}(Y,b)}) : \kappa_a \longrightarrow \kappa_b$, and analogously to Theorem 4.5 one may prove that $\mathcal{Q}$-**IntDist** is a retract and a coreflective subcategory of $\mathcal{Q}\text{-}\mathbf{Cat} \downarrow (-)^*$.

6. Diagonals between $\mathcal{Q}$-distributors as continuous $\mathcal{Q}$-distributors

For continuous $\mathcal{Q}$-distributors $\zeta, \zeta' : (X,a) \,{-\!\!\circ\!\!\!\rightarrow}\, (Y,b)$ between $\mathcal{Q}$-interior spaces, we denote by $\zeta \sim \zeta'$ if

$$\zeta^* b = \zeta'^* b. \tag{6.i}$$

To see the intuition of (6.i), let us consider the case that $\zeta = f_{\natural}$ and $\zeta' = f'_{\natural}$ for some continuous $\mathcal{Q}$-functors $f, f' : (X,a) \longrightarrow (Y,b)$. Then (6.i) becomes

$$f^{\leftarrow} b = f'^{\leftarrow} b;$$

that is, $f \sim f'$ if the preimages of each open presheaf under f and f' are identical.

It is not difficult to see that "$\sim$" gives rise to a congruence on the quantaloid $\mathcal{Q}$-**IntDist**, and we denote the induced quotient quantaloid by

$$(\mathcal{Q}\text{-}\mathbf{IntDist})_{\circ} := \mathcal{Q}\text{-}\mathbf{IntDist}/\sim.$$

Proposition 6.1. *For arrows $(\zeta,\eta), (\zeta',\eta') : (\varphi : X \,{-\!\!\circ\!\!\!\rightarrow}\, Y) \longrightarrow (\psi : X' \,{-\!\!\circ\!\!\!\rightarrow}\, Y')$ in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$, the following statements are equivalent:*

(i) $(\zeta,\eta) \sim (\zeta',\eta') : \varphi \longrightarrow \psi$.

(ii) $\zeta \sim \zeta' : (X, \varphi^*\varphi_*) \,{-\!\!\circ\!\!\!\rightarrow}\, (X', \psi^*\psi_*)$.

Proof. (i) $\Longrightarrow$ (ii): If $\psi \circ \zeta = \psi \circ \zeta'$, then the functoriality of $(-)^* : (\mathcal{Q}\text{-}\mathbf{Dist})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ ensures that $\zeta^*\psi^* = \zeta'^*\psi^*$. Thus $\zeta^*\psi^*\psi_* = \zeta'^*\psi^*\psi_*$.

(ii) $\Longrightarrow$ (i): If $\zeta^*\psi^*\psi_* = \zeta'^*\psi^*\psi_*$, then

$$\zeta^*\psi^* = \zeta^*\psi^*\psi_*\psi^* = \zeta'^*\psi^*\psi_*\psi^* = \zeta'^*\psi^*.$$

It follows from (3.ii) and (3.iii) that

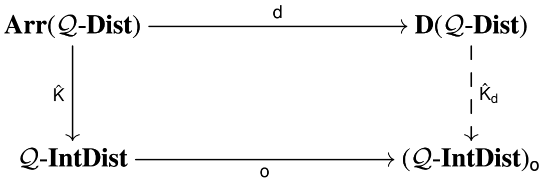
$$\psi(-,y') \circ \zeta = \zeta^*\psi^*\mathsf{y}_{Y'}y' = \zeta'^*\psi^*\mathsf{y}_{Y'}y' = \psi(-,y') \circ \zeta'$$

for all $y' \in Y'$. Thus $\psi \circ \zeta = \psi \circ \zeta'$. $\square$

Proposition 6.1 indicates that $\hat{\mathsf{K}}(\zeta,\eta) = \zeta : (X,\varphi^*\varphi_*) ⧴ (X',\psi^*\psi_*)$ is equal to $\hat{\mathsf{K}}(\zeta',\eta') = \zeta'$ in $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ whenever $(\zeta,\eta) \sim (\zeta',\eta') : \varphi \longrightarrow \psi$ in $\mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$. Hence, the universal property of the quotient quantaloid $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) = \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})/\sim$ ensures that there is a (unique) quantaloid homomorphism

$$\hat{\mathsf{K}}_{\mathsf{d}} : \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$$

making the square



commute, where d and o are the obvious quotient homomorphisms (so that the composition of o and $\hat{\mathsf{K}}$ is also a full quantaloid homomorphism).

Proposition 6.2. *For continuous $\mathcal{Q}$-distributors $\zeta,\zeta' : (X,a) ⧴ (Y,b)$ between $\mathcal{Q}$-interior spaces, the following statements are equivalent:*

(i) $\zeta \sim \zeta' : (X,a) ⧴ (Y,b)$.

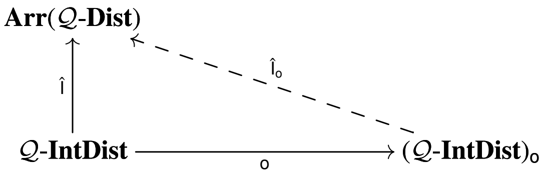
(ii) $(\zeta, (\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}) = (\zeta', (\zeta'^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}) : (\kappa_a : X ⧴ \mathcal{O}(X,a)) \longrightarrow (\kappa_b : Y ⧴ \mathcal{O}(Y,b)))$.

Proof. Just note that $(\zeta^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)} = (\zeta'^*)^\natural|_{\mathcal{O}(X,a),\mathcal{O}(Y,b)}$ means precisely $\zeta^*|_{\mathcal{O}(Y,b)} = \zeta'^*|_{\mathcal{O}(Y,b)}$, which is an alternative expression of (6.i). □

Proposition 6.2 shows that $\hat{\mathsf{I}}\zeta = \hat{\mathsf{I}}\zeta'$ whenever $\zeta \sim \zeta' : (X,a) ⧴ (Y,b)$. The universal property of the quotient quantaloid $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ then guarantees the existence of a (unique) quantaloid homomorphism

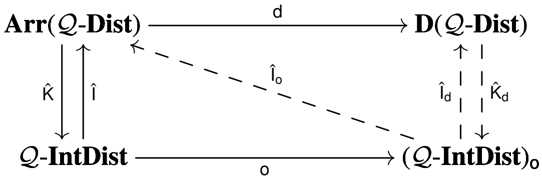
$$\hat{\mathsf{I}}_{\mathsf{o}} : (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \longrightarrow \mathbf{Arr}(\mathcal{Q}\text{-}\mathbf{Dist})$$

making the triangle



commute, and the composition of d and $\hat{\mathsf{I}}_{\mathsf{o}}$ produces a quantaloid homomorphism

$$\hat{\mathsf{I}}_{\mathsf{d}} : (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \longrightarrow \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}).$$



From Proposition 5.6 and the constructions of $\hat{\mathsf{K}}_{\mathsf{d}}$ and $\hat{\mathsf{I}}_{\mathsf{d}}$ it is easy to conclude:

Proposition 6.3. $\hat{\mathsf{K}}_{\mathsf{d}} : \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ *is a left inverse of* $\hat{\mathsf{I}}_{\mathsf{d}} : (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \longrightarrow \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$.

Note that Propositions 5.4 and 6.1 guarantee that $\hat{\mathsf{K}}_{\mathsf{d}}$ is fully faithful, and Proposition 6.3 implies that $\hat{\mathsf{K}}_{\mathsf{d}}$ is surjective on objects. Therefore, we arrive at the main result of this paper:

Theorem 6.4. $\hat{\mathsf{K}}_{\mathsf{d}} : \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \longrightarrow (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ *and* $\hat{\mathsf{I}}_{\mathsf{d}} : (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \longrightarrow \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ *establish an equivalence of quantaloids; hence,* $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ *and* $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ *are equivalent quantaloids.*

Proof. It remains to verify the claim about $\hat{\mathsf{I}}_{\mathsf{d}}$. Indeed, since $\hat{\mathsf{K}}_{\mathsf{d}}$ is an equivalence of quantaloids, there exists a functor $\mathsf{F} : (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \longrightarrow \mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ such that $\mathsf{F}\hat{\mathsf{K}}_{\mathsf{d}}$ is naturally isomorphic to the identity functor on $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$, thus so is $\hat{\mathsf{I}}_{\mathsf{d}}\hat{\mathsf{K}}_{\mathsf{d}}$ as there are natural isomorphisms

$$\hat{\mathsf{I}}_{\mathsf{d}}\hat{\mathsf{K}}_{\mathsf{d}} \cong \mathsf{F}\hat{\mathsf{K}}_{\mathsf{d}}\hat{\mathsf{I}}_{\mathsf{d}}\hat{\mathsf{K}}_{\mathsf{d}} \cong \mathsf{F}\hat{\mathsf{K}}_{\mathsf{d}},$$

showing that $\hat{\mathsf{I}}_{\mathsf{d}}$ is also an equivalence of quantaloids. □

Remark 6.5. It has been elaborated in [33, Subsection 1.1] that diagonals and back diagonals are dual constructions of each other. Their duality is once again supported by the equivalences of quantaloids

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}} \quad \text{and} \quad \mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq (\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}}$$

given by (1.iii) and Theorem 6.4, from the topological point of view:

- a diagonal between $\mathcal{Q}$-distributors is essentially an equivalence class of continuous $\mathcal{Q}$-distributors between $\mathcal{Q}$-interior spaces;
- a back diagonal between $\mathcal{Q}$-distributors is essentially an equivalence class of continuous $\mathcal{Q}$-distributors between $\mathcal{Q}$-closure spaces.

In the case that $\mathcal{Q} = \mathbf{2}$ is the two-element Boolean algebra, $\mathbf{D}(\mathbf{Dist})$ is the Freyd completion of the quantaloid **Dist** of (pre)ordered sets and distributors, while **IntDist** is the quantaloid of ordered interior spaces (i.e., ordered sets X equipped with an interior operator on its down-set lattice) and continuous distributors:

Corollary 6.6. *The Freyd completion* $\mathbf{D}(\mathbf{Dist})$ *of the quantaloid* **Dist** *is equivalent to* $\mathbf{IntDist}_{\mathsf{o}}$.

7. Diagonals between $\mathcal{Q}$-relations as continuous $\mathcal{Q}$-relations

Note that every set X over ob $\mathcal{Q}$ is equipped with a *discrete* $\mathcal{Q}$-category structure, given by

$$\mathrm{id}_X(x,y) = \begin{cases} 1_{|x|} & \text{if } x = y, \\ \bot_{|x|,|y|} & \text{else} \end{cases}$$

for all $x, y \in X$, where $\bot_{|x|,|y|}$ refers to the bottom element of the complete lattice $\mathcal{Q}(|x|,|y|)$. A $\mathcal{Q}$*-relation*

$$\varphi : X \to\!\!\!\!\!\mapsto Y$$

between sets over ob $\mathcal{Q}$ is precisely a $\mathcal{Q}$-distributor

$$\varphi : (X, \mathrm{id}_X) \;{\circ\!\!\!\!\longrightarrow}\; (Y, \mathrm{id}_Y).$$

Hence, sets over ob $\mathcal{Q}$ and $\mathcal{Q}$-relations constitute a full subquantaloid of $\mathcal{Q}$-**Dist**, denoted by

$$\mathcal{Q}\text{-}\mathbf{Rel}.$$

It is easy to see that a $\mathcal{Q}$-relation $\alpha : X \to\!\!\!\!\!\mapsto X$ defines a $\mathcal{Q}$-category (X, α) if

$$\mathrm{id}_X \leqslant \alpha \quad \text{and} \quad \alpha \circ \alpha \leqslant \alpha, \tag{7.i}$$

and a $\mathcal{Q}$-relation $\varphi : X \to\!\!\!\!\!\mapsto Y$ becomes a $\mathcal{Q}$-distributor $\varphi : (X,\alpha) \;{\circ\!\!\!\!\longrightarrow}\; (Y,\beta)$ if

$$\beta \circ \varphi \circ \alpha \leqslant \varphi. \tag{7.ii}$$

Proposition 7.1. $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$ *is equivalent to its full subquantaloid* $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Rel})$.

Proof. It suffices to show that every $\mathcal{Q}$-distributor $\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)$ is isomorphic to its underlying $\mathcal{Q}$-relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in the quantaloid $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$. Indeed, it is clear that the identity maps on X and Y are $\mathcal{Q}$-functorial as

$$1_X : (X,\mathrm{id}_X) \longrightarrow (X,\alpha) \quad \text{and} \quad 1_Y : (Y,\mathrm{id}_Y) \longrightarrow (Y,\beta).$$

It is routine to verify that

$$\begin{aligned}
&((1_X)_\natural,(1_Y)_\natural) : (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y) \longrightarrow (\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)) \quad \text{and}\\
&((1_X)^\natural,(1_Y)^\natural) : (\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)) \longrightarrow (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y)
\end{aligned}$$

$$\begin{array}{ccc}
(X,\mathrm{id}_X) & \xrightarrow[\circ]{(1_X)_\natural} & (X,\alpha)\\
\varphi\downarrow\circ & & \circ\downarrow\varphi\\
(Y,\mathrm{id}_Y) & \xrightarrow[(1_Y)_\natural]{\circ} & (Y,\beta)
\end{array}
\qquad
\begin{array}{ccc}
(X,\alpha) & \xrightarrow[\circ]{(1_X)^\natural} & (X,\mathrm{id}_X)\\
\varphi\downarrow\circ & & \circ\downarrow\varphi\\
(Y,\beta) & \xrightarrow[(1_Y)^\natural]{\circ} & (Y,\mathrm{id}_Y)
\end{array}$$

are arrows in $\mathcal{Q}$-**Dist**, and satisfy

$$\begin{aligned}
&((1_X)^\natural,(1_Y)^\natural) \circ ((1_X)_\natural,(1_Y)_\natural) = (\alpha,\beta) \sim (\mathrm{id}_X,\mathrm{id}_Y) : (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y) \longrightarrow (\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y),\\
&((1_X)_\natural,(1_Y)_\natural) \circ ((1_X)^\natural,(1_Y)^\natural) = (\alpha,\beta) : (\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)) \longrightarrow (\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)),
\end{aligned}$$

establishing an isomorphism between $\varphi : (X,\alpha) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,\beta)$ and $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist})$. □

Similarly, we denote by

$$\mathcal{Q}\text{-}\mathbf{IntRel} \quad \text{and} \quad (\mathcal{Q}\text{-}\mathbf{IntRel})_{\mathsf{o}}$$

the full subquantaloids of $\mathcal{Q}$-**IntDist** and $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$, respectively, whose objects are restricted to $\mathcal{Q}$-interior spaces (X,a) with X being discrete.

Proposition 7.2. *$(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$ is equivalent to its full subquantaloid $(\mathcal{Q}\text{-}\mathbf{IntRel})_{\mathsf{o}}$.*

Proof. Suppose that (X,α) is a $\mathcal{Q}$-category, i.e., $\alpha : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is a $\mathcal{Q}$-relation satisfying (7.i). If $a : \mathsf{P}(X,\alpha) \longrightarrow \mathsf{P}(X,\alpha)$ is a $\mathcal{Q}$-interior operator, then

$$a_0 : \mathsf{P}(X,\mathrm{id}_X) \longrightarrow \mathsf{P}(X,\mathrm{id}_X), \quad a_0\mu := a(\mu \swarrow \alpha)$$

defines a $\mathcal{Q}$-interior operator on $\mathsf{P}(X,\mathrm{id}_X)$. Indeed, $a_0 \leqslant 1_{\mathsf{P}(X,\mathrm{id}_X)}$ since

$$a_0\mu = a(\mu \swarrow \alpha) \leqslant \mu \swarrow \alpha \leqslant \mu \swarrow \mathrm{id}_X = \mu$$

for all $\mu \in \mathsf{P}(X,\mathrm{id}_X)$. As for $a_0 = a_0a_0$, note that for any $\mu \in \mathsf{P}(X,\mathrm{id}_X)$, $a_0\mu \in \mathsf{P}(X,\alpha)$ implies that $a_0\mu = a_0\mu \swarrow \alpha$, and $a_0\mu \in \mathcal{O}(X,\alpha,a)$ implies that $aa_0\mu = a_0\mu$. Thus

$$a_0\mu = aa_0\mu = a(a_0\mu \swarrow \alpha) = a_0a_0\mu.$$

Now it suffices to show that (X,α,a) is isomorphic to (X,id_X,a_0) in the quantaloid $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$. Note that

$$\mathcal{O}(X,\alpha,a) = \mathcal{O}(X,\mathrm{id}_X,a_0). \tag{7.iii}$$

Indeed, on one hand, $\mu \in \mathcal{O}(X,\alpha,a)$ implies that $a_0\mu = a(\mu \swarrow \alpha) = a\mu = \mu$, i.e., $\mu \in \mathcal{O}(X,\mathrm{id}_X,a_0)$. On the other hand, $\mu \in \mathcal{O}(X,\mathrm{id}_X,a_0)$ necessarily forces $\mu = a_0\mu = a(\mu \swarrow \alpha) \in \mathcal{O}(X,\alpha,a)$.

Since $1_X : (X,\mathrm{id}_X) \longrightarrow (X,\alpha)$ is a $\mathcal{Q}$-functor, its graph and cograph

$$(1_X)_\natural : (X,\mathrm{id}_X,a_0) \mathrel{-\!\!\circ\!\!\rightarrow} (X,\alpha,a), \quad (1_X)^\natural : (X,\alpha,a) \mathrel{-\!\!\circ\!\!\rightarrow} (X,\mathrm{id}_X,a_0)$$

are clearly continuous $\mathcal{Q}$-distributors by (7.iii), and satisfy

$$\begin{aligned}
&(1_X)^\natural \circ (1_X)_\natural = \alpha \sim \mathrm{id}_X : (X,\mathrm{id}_X,a_0) \mathrel{-\!\!\circ\!\!\rightarrow} (X,\mathrm{id}_X,a_0),\\
&(1_X)_\natural \circ (1_X)^\natural = \alpha : (X,\alpha,a) \mathrel{-\!\!\circ\!\!\rightarrow} (X,\alpha,a),
\end{aligned}$$

establishing an isomorphism between (X,α,a) and (X,id_X,a_0) in $(\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathsf{o}}$. □

From Theorem 6.4 and Propositions 7.1, 7.2 we soon deduce that:

Theorem 7.3. **D**($\mathcal{Q}$-**Rel**) *and* ($\mathcal{Q}$-**IntRel**)$_{\mathrm{o}}$ *are equivalent quantaloids.*

In the case that $\mathcal{Q} = \mathbf{2}$, **D**(**Rel**) is precisely the Freyd completion of the quantaloid **Rel** of sets and relations, while **IntRel** is the quantaloid of (classical) interior spaces (i.e., sets X equipped with an interior operator on its powerset $\mathbf{2}^X$) and continuous relations:

Corollary 7.4. *The Freyd completion* **D**(**Rel**) *of the quantaloid* **Rel** *is equivalent to* **IntRel**$_{\mathrm{o}}$.

Let **D**(**Rel**)$_{\mathrm{f}}$ denote the full subquantaloid of **D**(**Rel**) whose objects are relations $\varphi : X \to\!\!\!\!\!\!\!\!\;\;\;\circ\; Y$ such that $\widetilde{\varphi} : Y \longrightarrow \mathbf{2}^X$ is injective and that

$$\mathsf{Im}\,\widetilde{\varphi} = \{\widetilde{\varphi}y \mid y \in Y\} = \{\{x \in X \mid x\varphi y\} \mid y \in Y\}$$

is closed under arbitrary union and finite intersection. Then, by Remark 4.6, **D**(**Rel**)$_{\mathrm{f}}$ is clearly equivalent to the full subquantaloid **TopRel**$_{\mathrm{o}}$ of **IntRel**$_{\mathrm{o}}$ consisting of topological spaces:

Corollary 7.5. **D**(**Rel**)$_{\mathrm{f}}$ *and* **TopRel**$_{\mathrm{o}}$ *are equivalent quantaloids.*

8. When $\mathcal{Q}$ is a Girard quantaloid

Given a quantaloid $\mathcal{Q}$ and a family of $\mathcal{Q}$-arrows $\mathfrak{D} = \{d_q : q \longrightarrow q\}_{q \in \mathrm{ob}\,\mathcal{Q}}$, we say that

- $\mathfrak{D}$ is a *cyclic family*, if $d_p \swarrow u = u \searrow d_q$ for all $\mathcal{Q}$-arrows $u : p \longrightarrow q$;
- $\mathfrak{D}$ is a *dualizing family*, if $(d_p \swarrow u) \searrow d_p = u = d_q \swarrow (u \searrow d_q)$ for all $\mathcal{Q}$-arrows $u : p \longrightarrow q$.

$\mathcal{Q}$ is a *Girard quantaloid* [29] if it is equipped with a cyclic dualizing family of $\mathcal{Q}$-arrows. In this case, the *complement* of a $\mathcal{Q}$-arrow $u : p \longrightarrow q$ is defined as

$$\neg u = d_p \swarrow u = u \searrow d_q : q \longrightarrow p,$$

which clearly satisfies $\neg\neg u = u$, and it is straightforward to check that:

Proposition 8.1. *If* $\mathcal{Q}$ *is a Girard quantaloid, then*

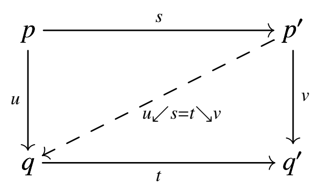
$$v \circ u = \neg(\neg u \swarrow v) = \neg(u \searrow \neg v)$$

for all $\mathcal{Q}$*-arrows* $u : p \longrightarrow q$, $v : q \longrightarrow r$.

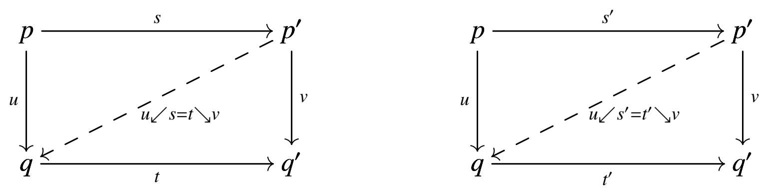
The aim of this section is to show that, in the case that $\mathcal{Q}$ is a small Girard quantaloid, we are able to concatenate the equivalences given by (1.iii) and Theorem 6.4.

Recall that each quantaloid $\mathcal{Q}$ induces a quantaloid **ChuCon**($\mathcal{Q}$), whose objects are $\mathcal{Q}$-arrows and whose morphisms are *Chu connections* [33] $(s,t) : (u : p \longrightarrow q) \longrightarrow (v : p' \longrightarrow q')$, i.e., pairs $(s : p \longrightarrow p',\ t : q \longrightarrow q')$ of $\mathcal{Q}$-arrows satisfying

$$u \swarrow s = t \searrow v.$$



For Chu connections $(s,t), (s',t') : u \longrightarrow v$, we denote by $(s,t) \sim (s',t')$ if the squares



generate the same *back diagonal*; that is, if

$$u \swarrow s = t \searrow v = u \swarrow s' = t' \searrow v.$$

"$\sim$" gives rise to a congruence on $\mathbf{ChuCon}(\mathcal{Q})$, and the induced quotient quantaloid, denoted by

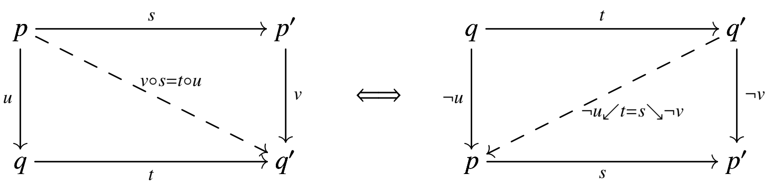
$$\mathbf{B}(\mathcal{Q}) := \mathbf{ChuCon}(\mathcal{Q})/\sim,$$

is called the quantaloid of *back diagonals* [33] of $\mathcal{Q}$.

Proposition 8.2. *If $\mathcal{Q}$ is a Girard quantaloid, then $\mathbf{Arr}(\mathcal{Q})$ and $\mathbf{ChuCon}(\mathcal{Q})$ are isomorphic quantaloids and, consequently, $\mathbf{D}(\mathcal{Q})$ and $\mathbf{B}(\mathcal{Q})$ are isomorphic quantaloids.*

Proof. Given $\mathcal{Q}$-arrows $u : p \longrightarrow q$, $v : p' \longrightarrow q'$ and a pair $(s : p \longrightarrow p',\ t : q \longrightarrow q')$ of $\mathcal{Q}$-arrows, it follows from Proposition 8.1 that

$$v \circ s = t \circ u \iff \neg u \swarrow t = s \searrow \neg v;$$



that is, $(s,t) : u \longrightarrow v$ is an arrow in $\mathbf{Arr}(\mathcal{Q})$ if, and only if, $(t,s) : \neg u \longrightarrow \neg v$ is a Chu connection. Hence, the assignment $((s,t) : u \longrightarrow v) \mapsto ((t,s) : \neg u \longrightarrow \neg v)$ defines an isomorphism of quantaloids

$$\neg : \mathbf{Arr}(\mathcal{Q}) \longrightarrow \mathbf{ChuCon}(\mathcal{Q})$$

which, clearly, also renders an isomorphism $\neg : \mathbf{D}(\mathcal{Q}) \longrightarrow \mathbf{B}(\mathcal{Q})$. □

Remark 8.3. The condition of $\mathcal{Q}$ being Girard is indispensable for the isomorphism $\mathbf{D}(\mathcal{Q}) \cong \mathbf{B}(\mathcal{Q})$. In the case that Q is a commutative quantale, it is already known from [19, Theorem 5.18] that there is an isomorphism $\mathbf{D}(\mathsf{Q}) \cong \mathbf{B}(\mathsf{Q})$ of quantaloids if, and only if, Q is a *Girard quantale* [28, 43], i.e., a one-object Girard quantaloid.

If $\mathcal{Q}$ is a small Girard quantaloid and X is a $\mathcal{Q}$-category, then

$$(\neg 1_X^{\natural})(y,x) = \neg 1_X^{\natural}(x,y)$$

defines a $\mathcal{Q}$-distributor $\neg 1_X^{\natural} : X {\circ\!\!\!\longrightarrow} X$, and it is straightforward to check that

$$\{\neg 1_X^{\natural} : X {\circ\!\!\!\longrightarrow} X\}_{X \in \mathrm{ob}(\mathcal{Q}\text{-}\mathbf{Dist})}$$

is a cyclic dualizing family of $\mathcal{Q}$-$\mathbf{Dist}$. In fact:

Proposition 8.4. [29] *A small quantaloid $\mathcal{Q}$ is Girard if, and only if, $\mathcal{Q}$-$\mathbf{Dist}$ is a Girard quantaloid.*

Therefore, Propositions 8.2 and 8.4 in conjunction with (1.iii) and Theorem 6.4 give rise to the following equivalences:

Theorem 8.5. *If $\mathcal{Q}$ is a small Girard quantaloid, then there are equivalences of quantaloids*

$$\mathbf{D}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq \mathbf{B}(\mathcal{Q}\text{-}\mathbf{Dist}) \simeq (\mathcal{Q}\text{-}\mathbf{IntDist})_{\mathrm{o}} \simeq (\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathrm{cl}} \simeq (\mathcal{Q}\text{-}\mathbf{Sup})^{\mathrm{op}}.$$

As a special case of Theorem 8.5, Corollary 7.4 actually amounts to the following equivalences in the classical case:

Corollary 8.6. *There are equivalences of quantaloids*

$$\mathbf{D}(\mathbf{Rel}) \simeq \mathbf{B}(\mathbf{Rel}) \simeq \mathbf{IntRel}_{\mathrm{o}} \simeq \mathbf{ClsRel}_{\mathrm{cl}} \simeq \mathbf{Sup}.$$

Remark 8.7. The equivalences $\mathbf{B}(\mathbf{Rel}) \simeq \mathbf{Sup}$ and $\mathbf{ClsRel}_{\mathrm{cl}} \simeq \mathbf{Sup}$ in Corollary 8.6 have appeared in [33, Corollary 3.4.5] and [32, Corollary 4.4.3], respectively, where $\mathbf{ClsRel}_{\mathrm{cl}}$ is the quantaloid of (classical) closure spaces (i.e., sets X equipped with a closure operator on its powerset $\mathbf{2}^X$) and closed continuous relations, and the self-duality of the quantaloid $\mathbf{Sup}$ of complete lattices and join-preserving maps is applied here.

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