On the probabilistic metrizability of approach spaces

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Abstract

We investigate approach spaces generated by probabilistic metric spaces with respect to a continuous t-norm $*$ on the unit interval $[0,1]$. Let k^* be the supremum of the idempotent elements of $*$ in $[0,1)$. It is shown that if $k^*=1$ (resp. $k^*<1$), then an approach space is probabilistic metrizable with respect to $*$ if and only if it is probabilistic metrizable with respect to the minimum (resp. product) t-norm.

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1. Introduction

Probabilistic metric spaces [20, 23] are a generalization of metric spaces in which the distance takes values in distance distributions instead of non-negative real numbers. *Approach spaces* [16, 17, 18], introduced by Lowen, are a common extension for topological spaces and metric spaces.

The motivation of this paper originates from Lawvere's presentation of metric spaces as enriched categories [15], and the discovery of Manes and Barr that topological spaces can be encoded in terms of ultrafilter convergence [19, 2], which give rise to the theory of *monoidal topology* [4, 5, 10]. Explicitly:

- Classical metric spaces (in the sense of Fréchet [7]) are symmetric, separated and finitary $[0,\infty]^{\mathrm{op}}$-categories, and probabilistic metric spaces are symmetric, separated and finitary Δ^+-categories [3, 9], where $[0,\infty]^{\mathrm{op}}$ is the Lawvere quantale [15, 22], and Δ^+ is the quantale of distance distributions.

- Classical topological spaces are **2**-valued topological spaces, and approach spaces are $[0,\infty]^{\mathrm{op}}$-valued topological spaces [14], where **2** is the two-element Boolean algebra.

Since the Lawvere quantale $[0,\infty]^{\mathrm{op}}=([0,\infty]^{\mathrm{op}},+,0)$ is isomorphic to the quantale $[0,1]=([0,1],\times,1)$ of the unit interval equipped with the usual product, the classical metrizability of topological spaces is actually to consider

- **2**-valued topological spaces generated by symmetric, separated and finitary $[0,1]$-categories.

So, it is natural to consider its counterpart by taking tensor products $\boxtimes$ in the category **Sup** of complete lattices and sup-preserving maps [11]; that is,

- $([0,\infty]^{\mathrm{op}}\boxtimes \mathbf{2})$-valued topological spaces generated by symmetric, separated and finitary $([0,\infty]^{\mathrm{op}}\boxtimes[0,1])$-categories.

Since $[0,\infty]^{\mathrm{op}}\boxtimes\mathbf{2}=[0,\infty]^{\mathrm{op}}$ (see [11, Proposition I.5.2]) and $[0,\infty]^{\mathrm{op}}\boxtimes[0,1]=\Delta^+$ (see [8, Subsection 3.3]), we are precisely asking for

- $[0,\infty]^{\mathrm{op}}$-valued topological spaces generated by symmetric, separated and finitary Δ^+-categories;

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that is,

- approach spaces generated by probabilistic metric spaces.

As a first step towards the probabilistic metrizability of approach spaces, in this paper we focus on the connections between approach spaces generated by probabilistic metric spaces with respect to different *continuous t-norms* [12, 1]. Explicitly, the triangle inequality in a probabilistic metric space (X, α) can be expressed as

$$\alpha(y, z, r) * \alpha(x, y, s) \leqslant \alpha(x, z, r + s)$$

for all $x, y, z \in X$, $r, s \in [0, \infty]$, where $*$ is a continuous t-norm[1] on the unit interval $[0, 1]$. Therefore, the probabilistic metrizability of an approach space relies on the choice of the continuous t-norm $*$. We say that an approach space is *probabilistic metrizable with respect to* $*$, or $*$*-metrizable* for short, if it is generated by a probabilistic metric with respect to a continuous t-norm $*$ on $[0, 1]$. The main result of this paper, Theorem 4.5, reveals that for each continuous t-norm $*$ on $[0, 1]$, the $*$-metrizability of approach spaces depends on the supremum of the idempotent elements of $*$ in $[0, 1)$ (denoted by k^*):

(1) If $k^* = 1$, then an approach space is $*$-metrizable if and only if it is probabilistic metrizable with respect to the *minimum t-norm* on $[0, 1]$.

(2) If $k^* < 1$, then an approach space is $*$-metrizable if and only if it is probabilistic metrizable with respect to the *product t-norm* on $[0, 1]$.

As examples, in Section 5 we prove the following:

- Metric approach spaces are $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ (Proposition 5.1).
- Approach spaces generated by metrizable topological spaces are $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ (Proposition 5.2).
- If an approach space is $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$, then it is uniform (Corollary 5.7).
- If an approach space is $*$-metrizable for some continuous t-norm $*$ on $[0, 1]$, then it is locally countable (Proposition 5.4).
- Approach spaces can be classified by their probabilistic metrizability (Proposition 5.10).

2. Preliminaries

Throughout this paper, let $[0, \infty]$ denote the extended non-negative real line, with the usual addition "+" and subtraction "−" extended via

$$t + \infty = \infty + t = \infty \quad \text{and} \quad \infty - t = \begin{cases} \infty & \text{if } t < \infty, \\ 0 & \text{if } t = \infty \end{cases}$$

to $[0, \infty]$. A map

$$\varphi \colon [0, \infty] \longrightarrow [0, 1]$$

is called a *distance distribution* (see [23, Definition 4.3.1]) if

(D1) $\varphi(0) = 0$, $\varphi(\infty) = 1$,

(D2) φ is monotone, and

(D3) φ is left-continuous on $(0, \infty)$.

[1] In some literature (e.g., [23]) the t-norm $*$ is only required to be left-continuous.

In particular, we have

$$\varphi(t) = \sup_{s<t} \varphi(s)$$

for all $t \in [0, \infty)$.

It is easy to verify the following lemma, which gives a canonical procedure of generating distance distributions:

Lemma 2.1. *For every monotone map $\varphi\colon [0,\infty] \longrightarrow [0,1]$, the map*

$$\psi\colon [0,\infty] \longrightarrow [0,1], \quad \psi(t) = \begin{cases} \sup\limits_{s<t} \varphi(s) & \text{if } t < \infty, \\ 1 & \text{if } t = \infty \end{cases}$$

is a distance distribution.

Recall that a binary operation $*$ on an interval $[a,b]$ is a *continuous t-norm* [12, 1] if

- $([a,b], *, b)$ is a commutative monoid,
- $p * q \leqslant p' * q'$ if $p \leqslant p'$ and $q \leqslant q'$ in $[a,b]$, and
- $*\colon [a,b]^2 \longrightarrow [a,b]$ is a continuous function (with respect to the usual topology).

It is obvious that $p * q \leqslant \min\{p,q\}$ for all $p,q \in [a,b]$; in particular, $a * q = a$ for all $q \in [a,b]$. An element $q \in [a,b]$ is *idempotent* if $q * q = q$ and, when it is non-idempotent, it necessarily holds that $q * q < q$.

Lemma 2.2. *Let $*$ be a continuous t-norm on $[a,b]$. If q is an idempotent element, then $p * q = \min\{p,q\}$ for all $p \in [a,b]$.*

Proof. If $p \geqslant q$, then $q = q * q \leqslant p * q$, and thus $p * q = q$. Suppose that $p \leqslant q$. Since $a * q = a$ and $q * q = q$, the continuity of $*$ guarantees the existence of $p' \in [a,q]$ such that $p' * q = p$. It follows that

$$p * q = p' * q * q = p' * q = p. \qquad \square$$

Given a continuous t-norm $*$ on $[a,b]$, define

$$q^- = \sup\{p \in [a,q] \mid p * p = p\} \tag{2.i}$$

for all $q \in [a,b]$. It is clear that q^- is the largest idempotent element in $[a,q]$.

Lemma 2.3. $(p * q)^- = \min\{p^-, q^-\}$ *for all $p,q \in [a,b]$.*

Proof. $(p * q)^- \leqslant \min\{p^-, q^-\}$ follows immediately from $p * q \leqslant \min\{p,q\}$. Conversely, by Lemma 2.2,

$$\min\{p^-, q^-\} = p^- * q^- \leqslant p * q,$$

and consequently $\min\{p^-, q^-\} \leqslant (p * q)^-$. $\square$

It is well known [6, 21, 12, 13, 1] that every continuous t-norm $*$ on the unit interval $[0,1]$ can be written as an *ordinal sum* of three basic t-norms, i.e., the minimum, the product, and the Łukasiewicz t-norm:

- The *minimum t-norm* $*_M$: $p *_M q = \min\{p,q\}$ for all $p,q \in [0,1]$.
- The *product t-norm* $*_P$: $p *_P q = pq$ for all $p,q \in [0,1]$.
- The *Łukasiewicz t-norm* $*_Ł$: $p *_Ł q = \max\{0, p+q-1\}$ for all $p,q \in [0,1]$.

Explicitly, we say that continuous t-norms $([a_1,b_1], *_1)$ and $([a_2,b_2], *_2)$ are isomorphic if there exists an order-isomorphism

$$\varphi\colon [a_1,b_1] \longrightarrow [a_2,b_2]$$

such that

$$\varphi\colon ([a_1,b_1], *_1, b_1) \longrightarrow ([a_2,b_2], *_2, b_2)$$

is an isomorphism of commutative monoids. Then:

Lemma 2.4. [12, 13, 1] *For each continuous t-norm* $([0,1],*)$*, the set of non-idempotent elements of* $*$ *in* $[0,1]$ *is a union of countably many pairwise disjoint open intervals*

$$\{(a_i,b_i) \mid 0 < a_i < b_i < 1,\ i \in I,\ I \text{ is countable}\},$$

and for each $i \in I$*, the continuous t-norm* $([a_i,b_i],*)$ *obtained by restricting* $*$ *to* $[a_i,b_i]$ *is either isomorphic to the product t-norm* $([0,1],*_P)$ *or isomorphic to the Łukasiewicz t-norm* $([0,1],*_Ł)$*.*

The *convolution* of distance distributions $\varphi,\psi\colon [0,\infty] \longrightarrow [0,1]$ with respect to a continuous t-norm $*$ on $[0,1]$ is defined as a map

$$\varphi \otimes_* \psi\colon [0,\infty] \longrightarrow [0,1], \quad (\varphi \otimes_* \psi)(t) = \begin{cases} \sup\limits_{r+s=t} \varphi(r) * \psi(s) & \text{if } t < \infty, \\ 1 & \text{if } t = \infty. \end{cases}$$

It is obvious that the distance distribution

$$\kappa\colon [0,\infty] \longrightarrow [0,1], \quad \kappa(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0 \end{cases}$$

is the neutral element for the convolution $\otimes_*$. A *probabilistic metric space* (see [23, Definition 8.1.1]) is a set X equipped with a map

$$\alpha\colon X \times X \times [0,\infty] \longrightarrow [0,1],$$

such that

(P1) $\alpha(x,y,-)\colon [0,\infty] \longrightarrow [0,1]$ is a distance distribution,

(P2) $\alpha(x,x,-) = \kappa$,

(P3) $\alpha(x,y,r) = \alpha(y,x,r)$,

(P4) $\alpha(x,y,-) = \kappa \implies x = y$, and

(P5) $\alpha(y,z,r) * \alpha(x,y,s) \leqslant \alpha(x,z,r+s)$

for all $x,y,z \in X$, $r,s \in [0,\infty]$, where the value

$$\alpha(x,y,r)$$

is interpreted as the probability that the distance between x and y is less than r. A map $f\colon (X,\alpha) \longrightarrow (Y,\beta)$ between probabilistic metric spaces is *non-expansive* if

$$\alpha(x,x',t) \leqslant \beta(f(x),f(x'),t)$$

for all $x,x' \in X$, $t \in [0,\infty]$. The category of probabilistic metric spaces (with respect to a continuous t-norm $*$ on $[0,1]$) and non-expansive maps is denoted by

$$\mathbf{ProbMet}_*.$$

Let **Met** denote the category of classical metric spaces (in the sense of Fréchet [7]) and non-expansive maps. There is a canonical embedding functor

$$\chi\colon \mathbf{Met} \longrightarrow \mathbf{ProbMet}_* \tag{2.ii}$$

that sends each metric space (X,d) to the probabilistic metric space (X,α_d) with

$$\alpha_d\colon X \times X \times [0,\infty] \longrightarrow [0,1], \quad \alpha_d(x,y,t) = \begin{cases} 0 & \text{if } t \leqslant d(x,y), \\ 1 & \text{if } t > d(x,y). \end{cases} \tag{2.iii}$$

Lemma 2.5. (1) *If* (X,α) *is a probabilistic metric space with respect to* $*_M$*, then* (X,α) *is also a probabilistic metric space with respect to every continuous t-norm* $*$ *on* $[0,1]$*.*

(2) *If* (X,α) *is a probabilistic metric space with respect to* $*_P$*, then* (X,α) *is also a probabilistic metric space with respect to* $*_Ł$.

Proof. Note that for any $p,q\in[0,1]$, we have $p*q\leqslant\min\{p,q\}$ and $p*_Ł q\leqslant pq$, where the latter follows from $pq-p*_Ł q=pq-p-q+1=(p-1)(q-1)\geqslant 0$. Hence, the conclusions follow from (P5). □

Let $\mathbf{2}^X$ denote the powerset of a set X. An *approach space* [16, 18] is a set X equipped with a map

$$\delta\colon X\times\mathbf{2}^X\longrightarrow[0,\infty],$$

such that

(A1) $\delta(x,\{x\})=0$,

(A2) $\delta(x,\varnothing)=\infty$,

(A3) $\delta(x,A\cup B)=\min\{\delta(x,A),\delta(x,B)\}$, and

(A4) $\delta(x,A)\leqslant\sup\limits_{b\in B}\delta(b,A)+\delta(x,B)$

for all $x\in X$, $A,B\in\mathbf{2}^X$. A map $f\colon(X,\delta)\longrightarrow(Y,\theta)$ between approach spaces is a *contraction* if

$$\delta(x,A)\geqslant\theta(f(x),f(A))$$

for all $x\in X$, $A\in\mathbf{2}^X$. We denote by

App

the category of approach spaces and contractions. It is well known (see, e.g., [18, Theorem 2.2.4]) that the category **Top** of topological spaces and continuous maps is a coreflective subcategory of **App**, with the coreflection

$$\mathsf{C}\colon\mathbf{App}\longrightarrow\mathbf{Top}\tag{2.iv}$$

sending each approach space (X,δ) to (X,cl_δ), where

$$\mathsf{cl}_\delta\colon\mathbf{2}^X\longrightarrow\mathbf{2}^X,\quad \mathsf{cl}_\delta(A)=\{x\in X\mid\delta(x,A)=0\}\tag{2.v}$$

is the topological closure operator induced by the approach structure δ.

3. Approach structures on probabilistic metric spaces

Let (X,α) be a probabilistic metric space with respect to a continuous t-norm $*$ on $[0,1]$. Define a map

$$\delta_\alpha\colon X\times\mathbf{2}^X\longrightarrow[0,\infty],\quad \delta_\alpha(x,A)=\inf\{r\in[0,\infty]\mid\sup_{a\in A}\alpha(x,a,r)=1\}.\tag{3.i}$$

Proposition 3.1. (X,δ_α) *is an approach space.*

Proof. (X,δ_α) clearly satisfies (A1) and (A2). For (A3) and (A4), let $x\in X$, $A,B\in\mathbf{2}^X$. First,

$$\begin{aligned}\delta_\alpha(x,A\cup B)&=\inf\{r\in[0,\infty]\mid\sup_{y\in A\cup B}\alpha(x,y,r)=1\}\\&=\inf\{r\in[0,\infty]\mid\sup_{a\in A}\alpha(x,a,r)=1\text{ or }\sup_{b\in B}\alpha(x,b,r)=1\}\\&=\min\{\inf\{r\in[0,\infty]\mid\sup_{a\in A}\alpha(x,a,r)=1\},\inf\{r\in[0,\infty]\mid\sup_{b\in B}\alpha(x,b,r)=1\}\}\\&=\min\{\delta_\alpha(x,A),\delta_\alpha(x,B)\}.\end{aligned}$$

Second, for each $t\in(0,\infty)$, we show that $\delta_\alpha(x,A)\leqslant t$ whenever $\sup\limits_{b\in B}\delta_\alpha(b,A)+\delta_\alpha(x,B)<t$. In this case, there exist $r,s\in[0,\infty)$ such that

- $r + s = t$,
- $\delta_\alpha(b, A) < r$ for all $b \in B$, and
- $\delta_\alpha(x, B) < s$.

For any $p \in (0,1)$, let $q \in (p,1)$. Then the continuity of $*$ guarantees the existence of $k \in (0,1)$ such that $p < q * k$. Note that $\delta_\alpha(x, B) < s$ and the monotonicity of $\alpha(x, b, -)$ guarantees that $\sup\limits_{b\in B} \alpha(x, b, s) = 1$. Thus there exists $b \in B$ such that $q < \alpha(x, b, s)$. Similarly, from $\sup\limits_{a\in A} \alpha(b, a, r) = 1$ we may find $a \in A$ such that $k < \alpha(b, a, r)$. It follows that

$$\alpha(x, a, t) \geqslant \alpha(b, a, r) * \alpha(x, b, s) \geqslant q * k > p,$$

and consequently $\sup\limits_{a\in A} \alpha(x, a, t) = 1$ by the arbitrariness of p. Hence $\delta_\alpha(x, A) \leqslant t$. □

Proposition 3.2. *Let $(X, \alpha), (Y, \beta)$ be probabilistic metric spaces with respect to a continuous t-norm $*$ on $[0,1]$. If $f\colon (X, \alpha) \longrightarrow (Y, \beta)$ is non-expansive, then $f\colon (X, \delta_\alpha) \longrightarrow (Y, \delta_\beta)$ is a contraction.*

Proof. Let $x \in X$ and $A \subseteq X$. For each $r \in [0, \infty]$, since $f\colon (X, \alpha) \longrightarrow (Y, \beta)$ is non-expansive, we have

$$\alpha(x, y, r) \leqslant \beta(f(x), f(y), r)$$

for each $y \in X$. Therefore,

$$\delta_\alpha(x, A) = \inf\{r \in [0, \infty] \mid \sup_{a\in A} \alpha(x, a, r) = 1\} \geqslant \inf\{r \in [0, \infty] \mid \sup_{a\in A} \beta(f(x), f(a), r) = 1\} = \delta_\beta(f(x), f(A)),$$

which implies that $f\colon (X, \delta_\alpha) \longrightarrow (Y, \delta_\beta)$ is a contraction. □

Thus, we obtain a faithful functor

$$\Gamma\colon \mathbf{ProbMet}_* \longrightarrow \mathbf{App}.$$

Note that the diagram

$$\begin{array}{ccc} \mathbf{Met} & \hookrightarrow & \mathbf{Top} \\ {\scriptstyle \chi}\downarrow & & \uparrow{\scriptstyle \mathsf{C}} \\ \mathbf{ProbMet}_* & \xrightarrow[\Gamma]{} & \mathbf{App} \end{array}$$

is commutative (see (2.ii) and (2.iv) for the definitions of χ and C). Indeed, given a classical metric space (X, d), it is easy to check that

$$\delta_{\alpha_d}(x, A) = d(x, A) := \inf_{a\in A} d(x, a)$$

for all $x \in X$, $A \subseteq X$; and consequently, the topological closure operator induced by δ_{α_d} (under the functor C) is precisely the closure operator induced by the metric d. Hence, the approach structure (3.i) on a probabilistic metric space is compatible with the metric topology on a classical metric space.

Moreover, for each probabilistic metric space (X, α), the topological space $\mathsf{C}\Gamma(X, \alpha)$ is exactly the *strong topology* (see [23, Definition 12.1.1]) for (X, α):

Proposition 3.3. *The neighborhood system $\{\mathcal{N}_x\}_{x\in X}$ of the topological space $\mathsf{C}\Gamma(X, \alpha)$ is given by*

$$\mathcal{N}_x = \{U_x(t) \mid t \in (0, \infty)\},$$

where

$$U_x(t) = \{y \in X \mid \alpha(x, y, t) > 1 - t\}.$$

Proof. By (2.v) and (3.i), the topological space $\mathsf{C}\Gamma(X,\alpha)$ is given by the closure operator

$$\mathsf{cl}_{\delta_\alpha}\colon \mathbf{2}^X \longrightarrow \mathbf{2}^X,$$

with

$$x \in \mathsf{cl}_{\delta_\alpha}(A) \iff \inf\{r \in [0,\infty] \mid \sup_{a\in A} \alpha(x,a,r) = 1\} = 0$$

for all $A \subseteq X$. Thus

$$\begin{aligned}
x \in \mathsf{cl}_{\delta_\alpha}(A) &\iff \forall r \in (0,\infty)\colon \sup_{a\in A} \alpha(x,a,r) = 1\\
&\iff \forall r \in (0,\infty),\ \forall t \in (0,r],\ \exists a \in A\colon \alpha(x,a,r) > 1-t\\
&\iff \forall t \in (0,\infty),\ \forall r \in [t,\infty),\ \exists a \in A\colon \alpha(x,a,r) > 1-t\\
&\iff \forall t \in (0,\infty),\ \exists a \in A\colon \alpha(x,a,t) > 1-t \qquad (\alpha(x,y,-) \text{ is monotone})\\
&\iff \forall t \in (0,\infty)\colon A \cap U_x(t) \neq \varnothing
\end{aligned}$$

for all $A \subseteq X$. Hence, the closure operator $\mathsf{cl}_{\delta_\alpha}$ and the neighborhood system $\{\mathcal{N}_x\}_{x\in X}$ generates the same topology. □

4. Probabilistic metrizability of approach spaces

Let $*$ be a continuous t-norm on $[0,1]$. We say that an approach space (X,δ) is *probabilistic metrizable with respect to* $*$, or $*$*-metrizable* for short, if there exists a probabilistic metric α with respect to $*$ on X, such that

$$\delta = \delta_\alpha.$$

Lemma 4.1. *Let $*$ be a continuous t-norm on $[0,1]$. Then every $*_M$-metrizable approach space is $*$-metrizable.*

Proof. This is an immediate consequence of Lemma 2.5(1). □

Lemma 4.2. *An approach space (X,δ) is $*_P$-metrizable if and only if it is $*_Ł$-metrizable.*

Proof. The necessity follows immediately from Lemma 2.5(2). For the sufficiency, let $\alpha\colon X\times X\times[0,\infty]\longrightarrow[0,1]$ be a probabilistic metric with respect to $*_Ł$ such that $\delta = \delta_\alpha$. Define a map

$$\beta\colon X\times X\times[0,\infty]\longrightarrow[0,1],\quad \beta(x,y,t) = \begin{cases} 0 & \text{if } t = 0,\\ e^{\alpha(x,y,t)-1} & \text{if } t > 0.\end{cases}$$

First, $\beta(x,y,-)$ is a distance distribution. Indeed, it is clear that $\beta(x,y,0) = 0$, $\beta(x,y,\infty) = 1$ and $\beta(x,y,-)$ is monotone. Since $\alpha(x,y,-)$ is left-continuous on $(0,\infty)$ and the exponential function is continuous, $\beta(x,y,-)$ is also left-continuous on $(0,\infty)$.

Second, β is a probabilistic metric with respect to $*_P$, since β clearly satisfies (P2), (P3) and (P4), while (P5) follows from

$$\beta(y,z,r) *_P \beta(x,y,s) = e^{\alpha(y,z,r)+\alpha(x,y,s)-2} \leqslant e^{\alpha(y,z,r)*_Ł\alpha(x,y,s)-1} \leqslant e^{\alpha(x,z,r+s)-1} = \beta(x,z,r+s)$$

for all $x,y,z \in X$, $r,s \in (0,\infty]$.

Finally, $\delta_\alpha = \delta_\beta$, because it is easy to see that

$$\sup_{a\in A} \alpha(x,a,r) = 1 \iff \sup_{a\in A} \beta(x,a,r) = 1$$

for all $x \in X$, $A \subseteq X$, $r \in [0,\infty]$. □

Let $*$ be a continuous t-norm on $[0,1]$. Define

$$k^* = \sup\{q \in [0,1) \mid q * q = q\};$$

that is, the supremum of the idempotent elements of $*$ in $[0,1)$.

Lemma 4.3. *Let $*$ be a continuous t-norm on* $[0,1]$ *with* $k^* = 1$. *Then*

$$\sup A = 1 \iff \sup\{a^- \mid a \in A\} = 1$$

for all $A \subseteq [0,1]$.

Proof. It suffices to prove the "only if" part. Suppose that $\sup A = 1$. For any $r < 1$, from $k^* = 1$ we may find an idempotent element q with $r < q < 1$, and from $\sup A = 1$ we may find $a \in A$ with $q < a \leqslant 1$. Thus $r < q \leqslant a^- \leqslant 1$. Since r is arbitrary, we deduce that $\sup\{a^- \mid a \in A\} = 1$. □

Lemma 4.4. *Let $*_1$ and $*_2$ be continuous t-norms on* $[0,1]$. *If there exists an idempotent element q of $*_1$ such that the continuous t-norms* $([q,1], *_1)$ *and* $([0,1], *_2)$ *are isomorphic, then an approach space* (X, δ) *is $*_1$-metrizable if and only if it is $*_2$-metrizable.*

Proof. Let $\varphi\colon ([q,1], *_1) \longrightarrow ([0,1], *_2)$ be an isomorphism of continuous t-norms. For the "only if" part, suppose that there is a probabilistic metric $\alpha\colon X \times X \times [0,\infty] \longrightarrow [0,1]$ with respect to $*_1$ such that $\delta = \delta_\alpha$. Define a map

$$\beta\colon X \times X \times [0,\infty] \longrightarrow [0,1], \quad \beta(x,y,t) = \begin{cases} 0 & \text{if } \alpha(x,y,t) < q, \\ \varphi(\alpha(x,y,t)) & \text{if } \alpha(x,y,t) \geqslant q. \end{cases}$$

We show that β is a probabilistic metric with respect to $*_2$ and $\delta_\beta = \delta$.

First, $\beta(x,y,-)$ is a distance distribution. It is clear that $\beta(x,y,0) = 0$, $\beta(x,y,\infty) = 1$ and $\beta(x,y,-)$ is monotone. For the left-continuity of $\beta(x,y,-)$, let $t \in [0,\infty)$.

- If $\alpha(x,y,t) < q$, then $\beta(x,y,t) = 0 = \sup\limits_{s<t} \beta(x,y,s)$.
- If $\alpha(x,y,t) = q$, then $\beta(x,y,t) = \varphi(q) = 0 = \sup\limits_{s<t} \beta(x,y,s)$.
- If $\alpha(x,y,t) > q$, then the left-continuity of $\alpha(x,y,-)$ guarantees that there exists $r < t$ such that $\alpha(x,y,s) > q$ for all $s \in (r,t)$. It follows that

$$\beta(x,y,t) = \varphi(\alpha(x,y,t)) = \varphi(\sup_{s\in(r,t)} \alpha(x,y,s)) = \sup_{s\in(r,t)} \varphi(\alpha(x,y,s)) = \sup_{s<t} \beta(x,y,s).$$

Second, β is a probabilistic metric with respect to $*_2$. It is clear that β satisfies (P2), (P3) and (P4). For (P5), let $x,y,z \in X$ and $r,s \in [0,\infty]$.

- If either $\alpha(y,z,r) < q$ or $\alpha(x,y,s) < q$, then

$$\beta(y,z,r) *_2 \beta(x,y,s) = 0 \leqslant \beta(x,z,r+s).$$

- If $\alpha(y,z,r) \geqslant q$ and $\alpha(x,y,s) \geqslant q$, then $\alpha(x,z,r+s) \geqslant q *_1 q = q$, and consequently

$$\begin{aligned} \beta(y,z,r) *_2 \beta(x,y,s) &= \varphi(\alpha(y,z,r)) *_2 \varphi(\alpha(x,y,s)) \\ &= \varphi(\alpha(y,z,r) *_1 \alpha(x,y,s)) \\ &\leqslant \varphi(\alpha(x,z,r+s)) \\ &= \beta(x,z,r+s). \end{aligned}$$

Finally, $\delta_\beta = \delta$. Since φ is an order-isomorphism, we have

$$\sup_{a\in A} \beta(x,a,r) = 1 \iff \sup_{a\in A} \alpha(x,a,r) = 1$$

for all $x \in X$, $A \subseteq X$, $r \in [0,\infty]$. The conclusion thus follows.

Conversely, for the "if" part, suppose that there is a probabilistic metric $\beta\colon X\times X\times[0,\infty]\longrightarrow[0,1]$ with respect to $*_2$ on X such that $\delta=\delta_\beta$. Define a map

$$\alpha\colon X\times X\times[0,\infty]\longrightarrow[0,1],\quad \alpha(x,y,t)=\begin{cases}0 & \text{if } t=0,\\ \varphi^{-1}(\beta(x,y,t)) & \text{if } t>0.\end{cases}$$

We show that α is a probabilistic metric with respect to $*_1$ and $\delta_\alpha=\delta$.

First, $\alpha(x,y,-)$ is a distance distribution, since $\alpha(x,y,0)=0$, $\alpha(x,y,\infty)=1$, and

$$\alpha(x,y,t)=\varphi^{-1}(\beta(x,y,t))=\varphi^{-1}(\sup_{s<t}\beta(x,y,s))=\sup_{s<t}\varphi^{-1}(\beta(x,y,s))=\sup_{s<t}\alpha(x,y,s)$$

for all $t\in(0,\infty)$.

Second, α is a probabilistic metric with respect to $*_1$, since α clearly satisfies (P2), (P3) and (P4), while (P5) follows from

$$\alpha(y,z,r)*_1\alpha(x,y,s)=\varphi^{-1}(\beta(y,z,r))*_1\varphi^{-1}(\beta(x,y,s))=\varphi^{-1}(\beta(y,z,r)*_2\beta(x,y,s))\leqslant\alpha(x,z,r+s),$$

for all $x,y,z\in X$, $r,s\in(0,\infty]$.

Finally, $\delta_\alpha=\delta$ follows immediately from the fact that

$$\sup_{a\in A}\alpha(x,a,r)=1\iff\sup_{a\in A}\beta(x,a,r)=1$$

for all $x\in X$, $A\subseteq X$, $r\in[0,\infty]$, which completes the proof. □

Theorem 4.5. *Let $*$ be a continuous t-norm on* $[0,1]$.

(1) *If $k^*=1$, then an approach space (X,δ) is $*$-metrizable if and only if it is $*_M$-metrizable.*

(2) *If $k^*<1$, then an approach space (X,δ) is $*$-metrizable if and only if it is $*_P$-metrizable.*

Proof. (1) The sufficiency is an immediate consequence of Lemma 4.1. Conversely, suppose that there is a probabilistic metric $\alpha\colon X\times X\times[0,\infty]\longrightarrow[0,1]$ with respect to $*$ on X such that $\delta=\delta_\alpha$. Define a map

$$\beta\colon X\times X\times[0,\infty]\longrightarrow[0,1],\quad \beta(x,y,t)=\begin{cases}\sup\limits_{s<t}\alpha(x,y,s)^- & \text{if } t<\infty,\\ 1 & \text{if } t=\infty.\end{cases}$$

We show that β is a probabilistic metric with respect to $*_M$ on X and $\delta=\delta_\beta$.

First, the monotonicity of the map $\alpha(x,y,-)^-\colon[0,\infty]\longrightarrow[0,1]$ and Lemma 2.1 guarantee that $\beta(x,y,-)$ is a distance distribution.

Second, β is a probabilistic metric with respect to $*_M$ on X, since β clearly satisfies (P2), (P3) and (P4), while (P5) follows from

$$\begin{aligned}\beta(y,z,r)*_M\beta(x,y,s)&=(\sup_{a<r}\alpha(y,z,a)^-)*_M(\sup_{b<s}\alpha(x,y,b)^-)\\&=\sup_{a<r,b<s}\min\{\alpha(y,z,a)^-,\alpha(x,y,b)^-\}\\&=\sup_{a<r,b<s}(\alpha(y,z,a)*\alpha(x,y,b))^- &&\text{(Lemma 2.3)}\\&\leqslant\sup_{a<r,b<s}\alpha(x,y,a+b)^-\\&\leqslant\sup_{t<r+s}\alpha(x,y,t)^-\\&=\beta(x,z,r+s)\end{aligned}$$

for all $x, y, z \in X$, $r, s \in [0, \infty)$.

Finally, $\delta = \delta_\beta$. Indeed,

$$\begin{aligned}
\delta_\beta(x, A) &= \inf\{r \in [0, \infty] \mid \sup_{a \in A} \beta(x, a, r) = 1\} \\
&= \inf\{r \in [0, \infty] \mid \sup_{a \in A} \sup_{s < r} \alpha(x, a, s)^- = 1\} \\
&= \inf\{r \in [0, \infty] \mid \sup_{a \in A} \sup_{s < r} \alpha(x, a, s) = 1\} && \text{(Lemma 4.3)} \\
&= \inf\{r \in [0, \infty] \mid \sup_{a \in A} \alpha(x, a, r) = 1\} \\
&= \delta(x, A)
\end{aligned}$$

for all $x \in X$, $A \subseteq X$, as desired.

(2) By Lemma 2.4, the continuous t-norm $([k^*, 1], *)$ is either isomorphic to $([0, 1], *_P)$ or $([0, 1], *_Ł)$.

- If $([k^*, 1], *)$ is isomorphic to $([0, 1], *_P)$, then it follows from Lemma 4.4 that (X, δ) is $*$-metrizable if and only if it is $*_P$-metrizable.
- If $([k^*, 1], *)$ is isomorphic to $([0, 1], *_Ł)$, then it follows from Lemmas 4.2 and 4.4 that (X, δ) is $*$-metrizable if and only if it is $*_P$-metrizable. □

5. Examples

5.1. Metric approach spaces

Given a (classical) metric space (X, d), the map

$$\delta_d\colon X \times 2^X \longrightarrow [0, \infty], \quad \delta_d(x, A) = \begin{cases} \inf_{y \in A} d(x, y) & \text{if } A \neq \varnothing, \\ \infty & \text{if } A = \varnothing \end{cases}$$

defines an approach structure on X. Such approach spaces are called *metric approach spaces* [17, 18].

Proposition 5.1. *Metric approach spaces are $*$-metrizable for every continuous t-norm $*$ on* $[0, 1]$.

Proof. Let (X, δ) be an approach space such that $\delta = \delta_d$ for some metric d on X. Let α_d be the probabilistic metric on X induced by d (see (2.iii)). Then the $*$-metrizability of (X, δ) follows from

$$\begin{aligned}
\delta_{\alpha_d}(x, A) &= \inf\{r \in [0, \infty] \mid \sup_{a \in A} \alpha_d(x, a, r) = 1\} \\
&= \inf\{r \in [0, \infty] \mid \exists\, a \in A\colon \alpha_d(x, a, r) = 1\} \\
&= \inf\{r \in [0, \infty] \mid \exists\, a \in A\colon r > d(x, a)\} \\
&= \inf_{a \in A} d(x, a) \\
&= \delta(x, A)
\end{aligned}$$

for all $x \in X$, $A \subseteq X$. □

5.2. Topological approach spaces

Given a topological space $(X, \mathcal{T})$, the map

$$\delta_{\mathcal{T}}\colon X \times 2^X \longrightarrow [0, \infty], \quad \delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & \text{if } x \text{ is in the closure of } A, \\ \infty & \text{otherwise} \end{cases}$$

defines an approach structure on X. Such approach spaces are called *topologically generated*.

Proposition 5.2. *If* $(X,\mathcal{T})$ *is a metrizable topological space, then* $(X,\delta_{\mathcal{T}})$ *is* $*$*-metrizable for every continuous t-norm* $*$ *on* $[0,1]$.

Proof. By Lemma 4.1, it suffices to show that $(X,\delta_{\mathcal{T}})$ is $*_M$-metrizable. Let d be a metric on X such that x is in the closure of A if and only if

$$\inf_{a\in A} d(x,a)=0$$

for all $x\in X$ and $A\subseteq X$. Let $\varphi_0=\kappa$, and for each $t\in(0,\infty)$, define a map

$$\varphi_t\colon [0,\infty]\longrightarrow[0,1],\quad \varphi_t(x)=\begin{cases}1-e^{-\frac{x}{t}} & \text{if } x\in[0,\infty),\\ 1 & \text{if } x=\infty.\end{cases}$$

Then φ_t is clearly a distance distribution for all $t\in[0,\infty)$. We claim that the map

$$\alpha\colon X\times X\times[0,\infty]\longrightarrow[0,1],\quad \alpha(x,y,r)=\varphi_{d(x,y)}(r)$$

is a probabilistic metric with respect to $*_M$ on X. Indeed, since d is a metric, α clearly satisfies (P2), (P3) and (P4). For (P5), we have to check that

$$\alpha(y,z,r)*_M\alpha(x,y,s)\leqslant\alpha(x,z,r+s) \tag{5.i}$$

for all $x,y,z\in X$ and $r,s\in[0,\infty)$. Note that (5.i) clearly holds when $d(y,z)=0$ or $d(x,y)=0$. Suppose that $d(y,z)\neq0$ and $d(x,y)\neq0$. Since

$$\min\Big\{\frac{r}{d(y,z)},\frac{s}{d(x,y)}\Big\}\leqslant\frac{r+s}{d(y,z)+d(x,y)}\leqslant\frac{r+s}{d(x,z)},$$

we have

$$\alpha(y,z,r)*_M\alpha(x,y,s)=\min\{1-e^{-\frac{r}{d(y,z)}},1-e^{-\frac{s}{d(x,y)}}\}\leqslant1-e^{-\frac{r+s}{d(x,z)}}=\alpha(x,z,r+s),$$

as desired. Finally, we show that

$$\delta_{\mathcal{T}}(x,A)=\delta_\alpha(x,A)$$

for all $x\in X$ and $A\subseteq X$. Let $p=\inf\limits_{a\in A}d(x,a)$. There are two cases:

- If $p>0$, then x is not in the closure of A, and thus $\delta_{\mathcal{T}}(x,A)=\infty$. On the other hand, since

  $$\sup_{a\in A}\alpha(x,a,r)=\sup_{a\in A}\varphi_{d(x,a)}(r)=\sup_{a\in A}(1-e^{-\frac{r}{d(x,a)}})=1-e^{-\frac{r}{p}}<1$$

  for all $r\in(0,\infty)$, we have $\delta_\alpha(x,A)=\inf\{r\in[0,\infty]\mid\sup\limits_{a\in A}\alpha(x,a,r)=1\}=\inf\{\infty\}=\infty$.
- If $p=0$, then $\delta_{\mathcal{T}}(x,A)=0$. On the other hand, it is easy to check that

  $$\sup_{a\in A}\alpha(x,a,r)=\sup_{a\in A}\varphi_{d(x,a)}(r)=1$$

  for all $r\in(0,\infty)$. Hence $\delta_\alpha(x,A)=\inf\{r\in[0,\infty]\mid\sup\limits_{a\in A}\alpha(x,a,r)=1\}=0$, which completes the proof. □

5.3. Locally countable approach spaces

Let (X,α) be a probabilistic metric space. For each $x\in X$ and $n\geqslant1$, define a map

$$\lambda_{x,n}\colon X\longrightarrow[0,\infty],\quad \lambda_{x,n}(y)=\inf\Big\{r\in[0,\infty]\;\Big|\;\alpha(x,y,r)>1-\frac{1}{n}\Big\}.$$

Lemma 5.3. *Let* (X,α) *be a probabilistic metric space. Then*

$$\delta_\alpha(x,A)=\sup_{n\geqslant1}\inf_{a\in A}\lambda_{x,n}(a)$$

for all $x\in X$, $A\subseteq X$.

Proof. For each $n \geqslant 1$ and $r \in [0, \infty]$, if $\sup_{a \in A} \alpha(x, a, r) = 1$, then there exists $a \in A$ such that $\alpha(x, a, r) > 1 - \frac{1}{n}$. It follows that

$$\inf_{a \in A} \lambda_{x,n}(a) = \inf_{a \in A} \inf \left\{ r \in [0, \infty] \;\middle|\; \alpha(x, a, r) > 1 - \frac{1}{n} \right\} \leqslant \inf\{ r \in [0, \infty] \mid \sup_{x \in A} \alpha(x, a, r) = 1\} = \delta_\alpha(x, A),$$

and consequently

$$\sup_{n \geqslant 1} \inf_{a \in A} \lambda_{x,n}(a) \leqslant \delta_\alpha(x, A).$$

Conversely, we show that for any $s \in (0, \infty)$, $\sup_{n \geqslant 1} \inf_{a \in A} \lambda_{x,n}(a) < s$ necessarily implies $\delta_\alpha(x, A) \leqslant s$. In this case, for any $n \geqslant 1$, we have

$$\inf_{a \in A} \lambda_{x,n}(a) = \inf_{a \in A} \inf \left\{ r \in [0, \infty] \;\middle|\; \alpha(x, a, r) > 1 - \frac{1}{n} \right\} < s,$$

which guarantees the existence of $a \in A$ such that $\alpha(x, a, s) > 1 - \frac{1}{n}$ (because $\alpha(x, a, -)$ is monotone). It follows that $\sup_{a \in A} \alpha(x, a, s) = 1$, and consequently

$$\delta_\alpha(x, A) = \inf\{ r \in [0, \infty] \mid \sup_{a \in A} \alpha(x, a, r) = 1\} \leqslant s.$$ □

Let (X, δ) be an approach space. For each $x \in X$, let

$$\mathcal{A}(x) = \{\varphi \colon X \longrightarrow [0, \infty] \mid \forall A \subseteq X \colon \inf_{a \in A} \varphi(a) \leqslant \delta(x, A)\}.$$

The collection $\{\mathcal{A}(x)\}_{x \in X}$ is called the associated *approach system* of (X, δ). A collection $\{\mathcal{B}(x)\}_{x \in X}$ is a *basis* for the approach system if

(B1) $\mathcal{B}(x) \subseteq \mathcal{A}(x)$ for all $x \in X$,

(B2) For any $\varphi_1, \varphi_2 \in \mathcal{B}(x)$, there exists $\varphi \in \mathcal{B}(x)$ such that $\varphi_1, \varphi_2 \leqslant \varphi$ (under the pointwise order).

(B3) Each $\varphi \in \mathcal{A}(x)$ is *dominated* by $\mathcal{B}(x)$ in the sense that

$$\forall \epsilon > 0,\ \forall \omega < \infty,\ \exists \psi \in \mathcal{B}(x),\ \forall y \in X \colon\ \min\{\varphi(y), \omega\} \leqslant \psi(y) + \epsilon.$$

(X, δ) is said to be *locally countable* [18] if the associated approach system $\{\mathcal{A}(x)\}_{x \in X}$ has a basis $\{\mathcal{B}(x)\}_{x \in X}$ such that each $\mathcal{B}(x)$ $(x \in X)$ is countable.

Proposition 5.4. *Every probabilistic metric space with respect to a continuous t-norm* $*$ *on* $[0, 1]$ *is locally countable.*

Proof. Let (X, α) be a probabilistic metric space and let $\mathcal{B}(x) = \{\lambda_{x,n} \mid n \geqslant 1\}$ for each $x \in X$. We show that $\{\mathcal{B}(x)\}_{x \in X}$ is a basis for $\{\mathcal{A}(x)\}_{x \in X}$, the associated approach system of (X, δ_α). Indeed, (B1) follows soon from Lemma 5.3, and (B2) holds since for any $n, m \geqslant 1$, $\lambda_{x,n}, \lambda_{x,m} \leqslant \lambda_{x,\max\{m,n\}}$ is an immediate consequence of the monotonicity of $\alpha(x, y, -)$ $(x, y \in X)$. It remains to prove (B3). We proceed by contradiction. Suppose that $\varphi \in \mathcal{A}(x)$ is not dominated by $\{\lambda_{x,n}\}_{n \geqslant 1}$. Then there exist $\epsilon > 0$, $\omega < \infty$ such that

$$A_n := \{a \in X \mid \min\{\varphi(a), \omega\} > \lambda_{x,n}(a) + \epsilon\} \neq \varnothing$$

for all $n \geqslant 1$. However, since $A_n \subseteq A_m$ whenever $m \leqslant n$, we have

$$\begin{aligned}
\sup_{n\geqslant 1} \delta_\alpha(x, A_n) + \epsilon &= \sup_{n\geqslant 1}\sup_{m\geqslant 1}\inf_{a\in A_n} \lambda_{x,m}(a) + \epsilon && \text{(Lemma 5.3)} \\
&\leqslant \sup_{n\geqslant 1}\sup_{m\geqslant 1}\inf_{a\in A_{\max\{n,m\}}} \lambda_{x,\max\{n,m\}}(a) + \epsilon && (A_{\max\{n,m\}} \subseteq A_n \text{ and } \lambda_{x,m} \leqslant \lambda_{x,\max\{n,m\}}) \\
&= \sup_{n\geqslant 1}\inf_{a\in A_n} \lambda_{x,n}(a) + \epsilon \\
&= \sup_{n\geqslant 1}\inf_{a\in A_n} (\lambda_{x,n}(a) + \epsilon) \\
&\leqslant \sup_{n\geqslant 1}\inf_{a\in A_n} \min\{\varphi(a), \omega\} && \text{(definition of } A_n) \\
&= \min\{\sup_{n\geqslant 1}\inf_{a\in A_n} \varphi(a), \omega\} \\
&\leqslant \min\{\sup_{n\geqslant 1} \delta_\alpha(x, A_n), \omega\}, && (\varphi \in \mathcal{A}(x))
\end{aligned}$$

which is a contradiction. □

As it is known from [18, Proposition 3.4.3] that a topologically generated approach space is locally countable if and only if its underlying topology is first-countable, the following corollary follows immediately from Proposition 5.4:

Corollary 5.5. *Let* (X, δ) *be a topologically generated approach space. If the underlying topology of* (X, δ) *is not first-countable, then* (X, δ) *is not* $*$*-metrizable for any continuous t-norm* $*$ *on* $[0, 1]$.

5.4. *Uniform approach spaces*

A *generalized metric* [15] (also *quasi-metric* [18]) on a set X is a map $d\colon X \times X \longrightarrow [0, \infty]$ satisfying

$$d(x, x) = 0 \quad \text{and} \quad d(y, z) + d(x, y) \geqslant d(x, z)$$

for all $x, y, z \in X$, and it is called *symmetric* if $d(x, y) = d(y, x)$ for all $x, y \in X$. We denote by

$$\mathcal{M}(X)$$

the family of all generalized metrics on X. Let $\rho \in \mathcal{M}(X)$ and $\mathcal{D} \subseteq \mathcal{M}(X)$. We say that ρ is *locally dominated by* $\mathcal{D}$ if

$$\forall x \in X,\ \forall \epsilon > 0,\ \forall \omega < \infty,\ \exists\, d \in \mathcal{D},\ \forall y \in X\colon\ \min\{\rho(x, y), \omega\} \leqslant d(x, y) + \epsilon,$$

and

$$\hat{\mathcal{D}} = \{\rho \in \mathcal{M}(X) \mid \rho \text{ is locally dominated by } \mathcal{D}\}$$

is called the *local saturation* of $\mathcal{D}$.

Let (X, δ) be an approach space, and let $\{\mathcal{A}(x)\}_{x\in X}$ be the associated approach system of (X, δ). The collection

$$\mathcal{G} = \{d \in \mathcal{M}(X) \mid \forall x \in X\colon d(x, -) \in \mathcal{A}(x)\}$$

is called the associated *gauge* of (X, δ). A collection $\mathcal{B} \subseteq \mathcal{G}$ is called a *basis* for the associated gauge of (X, δ) if

$$\mathcal{G} = \hat{\mathcal{B}}.$$

It is known that every generalized metric on X which is locally dominated by $\mathcal{G}$ belongs to $\mathcal{G}$ (see [18, Theorem 1.2.4]). An approach space is said to be *uniform* [18] if it has a gauge basis consisting of symmetric generalized metrics.

Proposition 5.6. *Every probabilistic metric space with respect to* $*_M$ *is a uniform approach space.*

Proof. Let (X, α) be a probabilistic metric space with respect to $*_M$. For each $n \geqslant 1$, define a map

$$d_n \colon X \times X \longrightarrow [0, \infty], \quad d_n(x, y) = \lambda_{x,n}(y) = \inf\Big\{r \in [0, \infty] \,\Big|\, \alpha(x, y, r) > 1 - \frac{1}{n}\Big\}.$$

It is clear that $d_n(x, x) = 0$ and $d_n(x, y) = d_n(y, x)$ for all $x, y \in X$. Let $x, y, z \in X$. If $r, s \in [0, \infty]$ satisfy

$$\alpha(y, z, s) > 1 - \frac{1}{n} \quad \text{and} \quad \alpha(x, y, r) > 1 - \frac{1}{n},$$

then

$$\alpha(x, z, r + s) \geqslant \alpha(y, z, s) *_M \alpha(x, y, r) > 1 - \frac{1}{n},$$

and consequently $d_n(y, z) + d_n(x, y) \geqslant d_n(x, z)$. Thus d_n is a symmetric generalized metric on X. We show that

$$\mathcal{B} = \{d_n \mid n \geqslant 1\}$$

is a basis for the associated gauge $\mathcal{G}$ of (X, δ_α). On one hand, $\hat{\mathcal{B}} \subseteq \mathcal{G}$ because it is clear that $d_n \in \mathcal{G}$ for all $n \geqslant 1$. On the other hand, let $\rho \in \mathcal{G}$. For any $x \in X$, $\epsilon > 0$, $\omega < \infty$, since $\rho(x, -)$ is dominated by $\mathcal{B}(x) = \{\lambda_{x,n} \mid n \geqslant 1\}$, there exists $n \geqslant 1$ such that

$$\min\{\rho(x, y), \omega\} \leqslant \lambda_{x,n}(y) + \epsilon = d_n(x, y) + \epsilon,$$

which means that ρ is locally dominated by $\mathcal{B}$, and consequently $\rho \in \hat{\mathcal{B}}$. Hence $\mathcal{G} \subseteq \hat{\mathcal{B}}$. □

Corollary 5.7. *If an approach space is $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$, then it is uniform.*

5.5. Classifications of approach spaces by probabilistic metrizability

The following two lemmas are easily verified with the aid of Theorem 4.5:

Lemma 5.8. *If an approach space (X, δ) is not $*_0$-metrizable for some continuous t-norm $*_0$ on $[0, 1]$, then it is not $*_M$-metrizable, and consequently not $*$-metrizable for any continuous t-norm $*$ on $[0, 1]$ with $k^* = 1$.*

Proof. Since (X, δ) is not $*_0$-metrizable, it follows from Lemma 4.1 that (X, δ) cannot be $*_M$-metrizable. Hence, by Theorem 4.5(1), (X, δ) is not $*$-metrizable for any continuous t-norm $*$ on $[0, 1]$ with $k^* = 1$. □

Lemma 5.9. *If an approach space (X, δ) is $*_0$-metrizable for some continuous t-norm $*_0$ on $[0, 1]$, then it is $*_P$-metrizable, and consequently $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ with $k^* < 1$.*

Proof. If $k^{*_0} = 1$, then (X, δ) is $*_M$-metrizable by Theorem 4.5(1), and thus $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ by Lemma 4.1.

If $k^{*_0} < 1$, then from Theorem 4.5(2) we deduce that (X, δ) is $*_P$-metrizable, and consequently $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ with $k^* < 1$. □

Therefore, we obtain the following classifications of approach spaces:

Proposition 5.10. *Let (X, δ) be an approach space. Then exactly one of the following three statements is true:*

1. *(X, δ) is $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$.*
2. *(X, δ) is $*_P$-metrizable, but not $*_M$-metrizable; in other words, (X, δ) is $*$-metrizable for every continuous t-norm $*$ on $[0, 1]$ with $k^* < 1$, but not $*$-metrizable for any continuous t-norm $*$ on $[0, 1]$ with $k^* = 1$.*
3. *(X, δ) is not $*$-metrizable for any continuous t-norm $*$ on $[0, 1]$.*

We have known that every metric approach space satisfies Proposition 5.10(1) (see Proposition 5.1), and every topologically generated approach space whose underlying topology is not first-countable satisfies Proposition 5.10(3) (see Corollary 5.5). However, it seems not easy to construct an approach space that satisfies 5.10(2). So, we end this paper with the following:

Question 5.11. Can we find an example of an approach space that is $*_P$-metrizable but not $*_M$-metrizable?

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