Q-**Set** is not generally a topos

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Abstract

For a commutative and divisible quantale Q, it is shown that the category of Q-sets is a topos if, and only if, Q is a frame.

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1. Introduction

The categorical foundation plays a crucial role in the study of fuzzy sets [30]. One of the most notable approaches is the theory of quantale-valued sets developed by Höhle and his collaborators [16, 10, 12, 13, 14, 15], which extends the theory of frame-valued sets initiated by Higgs [8, 9] and Fourman–Scott [5].

Explicitly, from every frame Ω we may construct a bicategory $\mathsf{D}\Omega$ [29], which is actually a *quantaloid* [25, 27, 28]. Symmetric categories enriched in the quantaloid $\mathsf{D}\Omega$ are precisely Ω-sets, and morphisms between Ω-sets are exactly left adjoint $\mathsf{D}\Omega$-distributors. It is well known that the category

$$\Omega\text{-}\mathbf{Set}$$

of Ω-sets and their morphisms is a topos (see [5, Theorem 5.9 and Proposition 9.2]).

If we consider an (involutive) *quantale* [24, 22] Q as the table of truth values, the theory of Q-sets can be established in the same way. More specifically, we may construct a quantaloid DQ [15, 23, 28] (see Proposition 4.2), and define Q-sets as symmetric DQ-categories [15, 23] (see Definition 4.1 and Proposition 4.3). Thus, we obtain the category

$$\mathsf{Q}\text{-}\mathbf{Set}$$

of Q-sets and left adjoint DQ-distributors. It is now natural to ask:

Question 1.1. Is Q-**Set** a topos?

This question was first proposed by Pu–Zhang (see [23, Question 7.2]) in the case that Q is a commutative and divisible quantale (also known as *GL-monoid* [10]), where they conjectured that Q-**Set** is a topos only when Q is a frame. The aim of this paper is to give an affirmative answer to their hypothesis (see Theorem 5.6):

- For a commutative and divisible quantale Q, the category Q-**Set** is a topos if, and only if, Q is a frame.

Therefore, the theory of topoi is unlikely to serve as the categorical foundation of Q-sets when the quantale Q is non-idempotent.

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2. Divisible quantales

A commutative and unital *quantale* [21, 24, 4] is a commutative monoid $(\mathsf{Q}, \&, 1)$ whose underlying set Q is a complete lattice (with a top element $\top$ and a bottom element $\bot$), such that

$$p \,\&\, \Big(\bigvee_{i\in I} q_i\Big) = \bigvee_{i\in I} p \,\&\, q_i$$

for all $p, q_i \in \mathsf{Q}$ $(i \in I)$. The right adjoint induced by the multiplication $\&$, denoted by $\rightarrow$,

$$(p \,\&\, -) \dashv (p \rightarrow -)\colon \mathsf{Q} \longrightarrow \mathsf{Q},$$

satisfies

$$p \,\&\, q \leqslant r \iff p \leqslant q \rightarrow r$$

for all $p, q, r \in \mathsf{Q}$. We say that Q is *divisible* if it satisfies one of the following equivalent conditions:

Proposition 2.1. *(See [23, Proposition 2.1].) The following statements are equivalent:*

(i) $u = q \,\&\, (q \rightarrow u)$ *whenever* $u \leqslant q$ *in* Q.

(ii) $v \,\&\, (q \rightarrow u) = (q \rightarrow v) \,\&\, u$ *whenever* $u, v \leqslant q$ *in* Q.

(iii) $u = q \,\&\, p$ *for some* $p \in \mathsf{Q}$ *whenever* $u \leqslant q$ *in* Q.

(iv) $p \wedge q = p \,\&\, (p \rightarrow q)$ *for all* $p, q \in \mathsf{Q}$.

In this case, the unit 1 *of the monoid* $(\mathsf{Q}, \&, 1)$ *must be the top element* $\top$ *of the complete lattice* Q.

Throughout this paper, we always assume that

$$(\mathsf{Q}, \&)$$

is a commutative and divisible quantale (also *GL-monoid* [10]).

Example 2.2. The following commutative and divisible quantales are well known:

(1) If Ω is a frame, then $(\Omega, \wedge)$ is a commutative and divisible quantale.

(2) If $*$ is a *continuous t-norm* [1, 17, 18] on the unit interval $[0, 1]$, then $([0, 1], *)$ is a commutative and divisible quantale.

(3) The Lawvere quantale $([0, \infty], +)$ [19] is commutative and divisible.

Let $q \in \mathsf{Q}$. We say that q is *idempotent* if $q \,\&\, q = q$. In this case, it is easy to see that

$$p \,\&\, q = p \wedge q$$

for all $p \in \mathsf{Q}$.

Proposition 2.3. *(See [11, Theorem 5.2].) The underlying complete lattice of* Q *is a frame. In particular, the idempotent elements of* Q *form a subframe of the frame* $(\mathsf{Q}, \wedge)$.

By Proposition 2.1(ii), it makes sense to define

$$v \circ_q u := v \,\&\, (q \rightarrow u) = (q \rightarrow v) \,\&\, u \tag{2.i}$$

whenever $u, v \leqslant q$ in Q.

Proposition 2.4. *(See [23, Proposition 2.5].) For every* $q \in \mathsf{Q}$*,* $({\downarrow}\, q, \circ_q)$ *is a commutative and divisible quantale, where* ${\downarrow}\, q = \{p \in \mathsf{Q} \mid p \leqslant q\}$.

Moreover, we say that $p \in \mathsf{Q}$ is *idempotent relative to* $q \in \mathsf{Q}$ [23] if p is idempotent in the quantale $(\downarrow q, \circ_q)$; that is, if

$$p \leqslant q \quad \text{and} \quad p \circ_q p = p \,\&\, (q \to p) = p. \tag{2.ii}$$

Hence, by Propositions 2.3 and 2.4, the set

$$C_q := \{p \leqslant q \mid p \,\&\, (q \to p) = p\} \tag{2.iii}$$

of idempotent elements relative to q is a subframe of the underlying frame of the quantale $(\downarrow q, \circ_q)$.

Proposition 2.5. *Suppose that $q \in \mathsf{Q}$ and $p \in C_q$. Then $p \wedge r = p \,\&\, (q \to r)$ for all $r \in \mathsf{Q}$.*

Proof. On one hand, since 1 is the top element of Q and $p \in C_q$, it is clear that

$$p \,\&\, (q \to r) \leqslant p \,\&\, 1 = p \quad \text{and} \quad p \,\&\, (q \to r) \leqslant q \,\&\, (q \to r) \leqslant r.$$

On the other hand,

$$p \wedge r = p \,\&\, (p \to r) = p \,\&\, (q \to p) \,\&\, (p \to r) \leqslant p \,\&\, (q \to r),$$

where the first equality follows from Proposition 2.1(iv), and the second equality follows from $p \in C_q$ and (2.iii). □

For any $p, q \in \mathsf{Q}$, define

$$p \sqsubseteq q \iff p \in C_q \iff p \leqslant q \text{ and } p \,\&\, (q \to p) = p. \tag{2.iv}$$

Proposition 2.6. *$\sqsubseteq$ is a partial order on Q. Therefore, C_q is the principal $\sqsubseteq$-lower set generated by $q \in \mathsf{Q}$.*

Proof. $\sqsubseteq$ is clearly reflexive and antisymmetric. For the transitivity, suppose that $p \sqsubseteq q$ and $q \sqsubseteq r$. Then

$$\begin{aligned}
p &= q \,\&\, (q \to p) && (p \leqslant q \text{ and Proposition 2.1(i)})\\
&= q \,\&\, (r \to q) \,\&\, (q \to p) && (q \sqsubseteq r \text{ and (2.iv)})\\
&= (r \to q) \,\&\, p && (p \leqslant q \text{ and Proposition 2.1(i)})\\
&= (r \to q) \,\&\, (q \to p) \,\&\, p && (p \sqsubseteq q \text{ and (2.iv)})\\
&\leqslant p \,\&\, (r \to p).
\end{aligned}$$

Since the reverse inequality is trivial, we conclude that $p \,\&\, (r \to p) = p$. Hence $p \sqsubseteq r$. □

Lemma 2.7. *If $p \leqslant q \leqslant r$ and $p \sqsubseteq r$, then $p \sqsubseteq q$.*

Proof. This is an immediate consequence of $p = p \,\&\, (r \to p) \leqslant p \,\&\, (q \to p)$. □

Lemma 2.8. *If $A \subseteq \mathsf{Q}$ and $a \sqsubseteq q$ for all $a \in A$, then $\bigvee A \sqsubseteq q$.*

Proof. Since $a \sqsubseteq q$ for all $a \in A$, by (2.iv) we have $\bigvee A \leqslant q$ and

$$\bigvee A = \bigvee_{a \in A} a \,\&\, (q \to a) \leqslant \bigvee_{a \in A} a \,\&\, (q \to \bigvee A) = (\bigvee A) \,\&\, (q \to \bigvee A).$$

Thus $\bigvee A \sqsubseteq q$. □

Proposition 2.9. *$(\mathsf{Q}, \sqsubseteq)$ is a complete lattice.*

Proof. Let $A \subseteq \mathsf{Q}$. Since it is clear that $\bot \sqsubseteq q$ for all $q \in \mathsf{Q}$, A has $\bot$ as its $\sqsubseteq$-lower bound. We show that

$$\sqcap A := \bigvee \{q \in \mathsf{Q} \mid \forall a \in A : q \sqsubseteq a\} \tag{2.v}$$

is the $\sqsubseteq$-meet of A. First, $\sqcap A$ is a $\sqsubseteq$-lower bound of A by Lemma 2.8. Second, if $q \sqsubseteq a$ for all $a \in A$, then $q \leqslant \sqcap A \leqslant a$ for all $a \in A$, and consequently $q \sqsubseteq \sqcap A$ by by Lemma 2.7. □

Note that the $\sqsubseteq$-join $\sqcup A$ of $A \subseteq \mathsf{Q}$ coincides with its $\leqslant$-join:

Proposition 2.10. *$\bigvee A = \sqcup A$ for all $A \subseteq \mathsf{Q}$.*

Proof. It is clear that $a \leqslant \bigvee A \leqslant \sqcup A$ for all $a \in A$. Thus, by Lemma 2.7, we have $a \sqsubseteq \bigvee A$ for all $a \in A$, which means that $\sqcup A \leqslant \bigvee A$. Hence $\bigvee A = \sqcup A$. □

3. Quantaloid-enriched categories and their Cauchy completeness

A *quantaloid* [25] $\mathcal{Q}$ is a category whose hom-sets are complete lattices, such that the composition $\circ$ of $\mathcal{Q}$-arrows preserves suprema on both sides, i.e.,

$$v\circ\Big(\bigvee_{i\in I}u_i\Big)=\bigvee_{i\in I}v\circ u_i\quad\text{and}\quad\Big(\bigvee_{i\in I}v_i\Big)\circ u=\bigvee_{i\in I}v_i\circ u$$

for all $\mathcal{Q}$-arrows $u,u_i\colon p\longrightarrow q$, $v,v_i\colon q\longrightarrow r$ $(i\in I)$. The corresponding right adjoints induced by the composition maps

$$(-\circ u)\dashv(-\swarrow u)\colon \mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(q,r)\quad\text{and}\quad(v\circ -)\dashv(v\searrow -)\colon\mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(p,q)$$

satisfy

$$v\circ u\leqslant w\iff v\leqslant w\swarrow u\iff u\leqslant v\searrow w$$

for all $\mathcal{Q}$-arrows $u\colon p\longrightarrow q$, $v\colon q\longrightarrow r$, $w\colon p\longrightarrow r$.

If a pair of $\mathcal{Q}$-arrows $u\colon p\longrightarrow q$ and $v\colon q\longrightarrow p$ satisfy

$$1_p\leqslant v\circ u\quad\text{and}\quad u\circ v\leqslant 1_q,$$

we say that u and v form an adjunction in $\mathcal{Q}$ and denote it by $u\dashv v$, where u is called a *left adjoint* of v, and v is a *right adjoint* of u. In this case, it is easy to check that

$$v=u\searrow 1_q\quad\text{and}\quad u=1_q\swarrow v.$$

Therefore, the right adjoint of a $\mathcal{Q}$-arrow, when it exists, is necessarily unique. We define

$$u^*:=u\searrow 1_q\colon q\longrightarrow p \tag{3.i}$$

for each $\mathcal{Q}$-arrow $u\colon p\longrightarrow q$.

Lemma 3.1. *(See [6, Proposition 2.3.4].) If $u\colon p\longrightarrow q$ is a left adjoint in $\mathcal{Q}$, then*

$$v\circ u=v\swarrow u^*\quad\textit{and}\quad u^*\circ w=u\searrow w$$

for all $\mathcal{Q}$-arrows $v\colon q\longrightarrow r$, $w\colon r\longrightarrow q$.

We say that $\mathcal{Q}$ is *involutive* if $\mathcal{Q}$ is equipped with an *involution*

$$(-)^\circ\colon\mathcal{Q}^{\mathrm{op}}\longrightarrow\mathcal{Q},$$

which is a functor satisfying

$$q^\circ=q,\quad u^{\circ\circ}=u\quad\text{and}\quad\Big(\bigvee_{i\in I}u_i\Big)^\circ=\bigvee_{i\in I}u_i^\circ$$

for all $q\in\mathrm{ob}\,\mathcal{Q}$ and $\mathcal{Q}$-arrows $u,u_i\colon p\longrightarrow q$ $(i\in I)$.

Given a small (i.e., $\mathrm{ob}\,\mathcal{Q}$ is a set) and involutive quantaloid $\mathcal{Q}$, a *$\mathcal{Q}$-category* (also *category enriched in* $\mathcal{Q}$) [27] consists of a set X, a *type* map $|\text{-}|:X\longrightarrow\mathrm{ob}\,\mathcal{Q}$ and a family of $\mathcal{Q}$-arrows $\alpha(x,y)\in\mathcal{Q}(|x|,|y|)$ $(x,y\in X)$, such that

$$1_{|x|}\leqslant\alpha(x,x)\quad\text{and}\quad\alpha(y,z)\circ\alpha(x,y)\leqslant\alpha(x,z)$$

for all $x,y,z\in X$. Note that (X,α) has an underlying (pre)order given by

$$x\leqslant y\iff |x|=|y|\text{ and }1_{|x|}\leqslant\alpha(x,y).$$

We say that (X,α) is *separated* (or *skeletal*) if the underlying order on X is a partial order. Moreover, (X,α) is *symmetric* [7] if

$$\alpha(x,y)=\alpha(y,x)^\circ \tag{3.ii}$$

for all $x, y \in X$.

A $\mathcal{Q}$-*functor* $f : (X,\alpha) \longrightarrow (Y,\beta)$ between $\mathcal{Q}$-categories is a map $f : X \longrightarrow Y$ such that

$$|x| = |fx| \quad \text{and} \quad \alpha(x,y) \leqslant \beta(fx, fy) \tag{3.iii}$$

for all $x, y \in X$. $\mathcal{Q}$-categories and $\mathcal{Q}$-functors constitute a category

$$\mathcal{Q}\text{-}\mathbf{Cat}.$$

A $\mathcal{Q}$-*distributor* $\varphi\colon (X,\alpha) ⇸ (Y,\beta)$ between $\mathcal{Q}$-categories is a map that assigns to each pair $(x,y) \in X \times Y$ a $\mathcal{Q}$-arrow $\varphi(x,y) \in \mathcal{Q}(|x|,|y|)$, such that

$$\beta(y,y') \circ \varphi(x,y) \circ \alpha(x',x) \leqslant \varphi(x',y')$$

for all $x, x' \in X$, $y, y' \in Y$. With the pointwise order inherited from $\mathcal{Q}$, the category

$$\mathcal{Q}\text{-}\mathbf{Dist}$$

of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors becomes a (large) quantaloid in which

$$\psi \circ \varphi\colon (X,\alpha) ⇸ (Z,\gamma), \quad (\psi \circ \varphi)(x,z) = \bigvee_{y\in Y} \psi(y,z) \circ \varphi(x,y),$$
$$\xi \swarrow \varphi\colon (Y,\beta) ⇸ (Z,\gamma), \quad (\xi \swarrow \varphi)(y,z) = \bigwedge_{x\in X} \xi(x,z) \swarrow \varphi(x,y),$$
$$\psi \searrow \xi\colon (X,\alpha) ⇸ (Y,\beta), \quad (\psi \searrow \xi)(x,y) = \bigwedge_{z\in Z} \psi(y,z) \searrow \xi(x,z)$$

for all $\mathcal{Q}$-distributors $\varphi\colon (X,\alpha) ⇸ (Y,\beta)$, $\psi\colon (Y,\beta) ⇸ (Z,\gamma)$, $\xi\colon (X,\alpha) ⇸ (Z,\gamma)$; the identity $\mathcal{Q}$-distributor on (X,α) is given by its hom $\alpha\colon (X,\alpha) ⇸ (X,\alpha)$.

Adjoint $\mathcal{Q}$-distributors are precisely adjunctions in the quantaloid $\mathcal{Q}$-**Dist**. Each $\mathcal{Q}$-functor $f\colon (X,\alpha) \longrightarrow (Y,\beta)$ induces a pair of adjoint $\mathcal{Q}$-distributors $f_\natural \dashv f^*_\natural$, given by

$$f_\natural\colon (X,\alpha) ⇸ (Y,\beta), \quad f_\natural(x,y) = \beta(fx,y) \quad \text{and} \quad f^*_\natural\colon (Y,\beta) ⇸ (X,\alpha), \quad f^*_\natural(y,x) = \beta(y,fx).$$

It is straightforward to verify the following lemma:

Lemma 3.2. *A $\mathcal{Q}$-functor $f\colon (X,\alpha) \longrightarrow (Y,\beta)$ is* fully faithful *in the sense that*

$$\alpha(x,x') = \beta(fx, fx')$$

for all $x, x' \in X$ if, and only if, $f^*_\natural \circ f_\natural = \alpha$.

For each $q \in \operatorname{ob}\mathcal{Q}$, let $\{q\}$ denote the (necessarily symmetric) one-object $\mathcal{Q}$-category whose only object has type q and hom 1_q. For every $\mathcal{Q}$-category (X,α), it is easy to see that

$$\alpha(x,-) \dashv \alpha(-,x)\colon (X,\alpha) ⇸ \{|x|\} \tag{3.iv}$$

for all $x \in X$. In fact, considering the $\mathcal{Q}$-functor

$$x\colon \{|x|\} \longrightarrow (X,\alpha) \tag{3.v}$$

targeting at x, it is clear that $\alpha(x,-) = x_\natural$ and $\alpha(-,x) = x^*_\natural$.

A $\mathcal{Q}$-category (X,α) is *Cauchy complete* if every left adjoint $\mathcal{Q}$-distributor

$$\mu\colon \{q\} ⇸ (X,\alpha)$$

is *representable*; that is, there exists $x \in X$, called the *representation* of μ, such that $\mu = \alpha(x, -)$.

Let (X, α) be a $\mathcal{Q}$-category. Define

$$\widehat{X} := \{\mu\colon \{q\} \to\!\!\!\!\!\circ\!\!\to (X, \alpha) \mid \mu \text{ is a left adjoint in } \mathcal{Q}\text{-}\mathbf{Dist},\ q \in \mathrm{ob}\,\mathcal{Q}\} \tag{3.vi}$$

and

$$\widehat{\alpha}(\mu, \lambda) := \lambda \searrow \mu = \lambda^* \circ \mu = (\lambda \searrow \alpha) \circ \mu \tag{3.vii}$$

for all $\mu, \lambda \in \widehat{X}$, where the last two equalities follow from Lemma 3.1 and Equation (3.i), respectively. Then $(\widehat{X}, \widehat{\alpha})$ is a separated and Cauchy complete $\mathcal{Q}$-category, called the *Cauchy completion* of (X, α) (cf. [2] and [23, Proposition 4.12]). There is a fully faithful $\mathcal{Q}$-functor

$$\mathfrak{y}\colon (X, \alpha) \longrightarrow (\widehat{X}, \widehat{\alpha}), \quad \mathfrak{y}x = \alpha(x, -), \tag{3.viii}$$

which is also injective when (X, α) is separated. Moreover, the assignment $(X, \alpha) \mapsto (\widehat{X}, \widehat{\alpha})$ is functorial; that is, there is a functor

$$\widehat{(-)}\colon \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{CcCat} \tag{3.ix}$$

sending each $\mathcal{Q}$-functor $f\colon (X, \alpha) \longrightarrow (Y, \beta)$ to

$$\widehat{f}\colon (\widehat{X}, \widehat{\alpha}) \longrightarrow (\widehat{Y}, \widehat{\beta}), \quad \widehat{f}\mu = f_\natural \circ \mu, \tag{3.x}$$

where $\mathcal{Q}$-**CcCat** is the full reflective subcategory of $\mathcal{Q}$-**Cat** consisting of separated and Cauchy complete $\mathcal{Q}$-categories (see [27, Proposition 7.14]). Hence, there is a bijection

$$\mathcal{Q}\text{-}\mathbf{CcCat}((\widehat{X}, \widehat{\alpha}), (Y, \beta)) \cong \mathcal{Q}\text{-}\mathbf{Cat}((X, \alpha), (Y, \beta)) \tag{3.xi}$$

natural in $(X, \alpha) \in \mathcal{Q}$-**Cat** and $(Y, \beta) \in \mathcal{Q}$-**CcCat**. In particular, the transpose of a $\mathcal{Q}$-functor $f\colon (X, \alpha) \longrightarrow (Y, \beta)$ is

$$\overline{f}\colon (\widehat{X}, \widehat{\alpha}) \longrightarrow (Y, \beta), \tag{3.xii}$$

with $\overline{f}\mu$ being the representation of $\widehat{f}\mu$ (see (3.x)) for all $\mu \in \widehat{X}$.

In this paper we are concerned with the full subcategory

$$\mathcal{Q}\text{-}\mathbf{CcSymCat}$$

of $\mathcal{Q}$-**Cat**, whose objects are separated, symmetric and Cauchy complete $\mathcal{Q}$-categories. For later use, we point out that equalizers in $\mathcal{Q}$-**CcSymCat** are formulated in the same way as in $\mathcal{Q}$-**Cat** (cf. [26, Remark 2.4(3)]):

Proposition 3.3. *The equalizer of $f, g\colon (X, \alpha) \longrightarrow (Y, \beta)$ in $\mathcal{Q}$-**CcSymCat** is given by*

$$E = \{x \in X \mid fx = gx\}$$

equipped with the restriction of the $\mathcal{Q}$-category structure α on E.

Proof. Let $\sigma = \alpha|(E \times E)$. By [26, Remark 2.4(3)], it suffices to show that (E, σ) is Cauchy complete. Let $\mu\colon \{q\} \to\!\!\!\!\!\circ\!\!\to (E, \sigma)$ be a left adjoint $\mathcal{Q}$-distributor. Considering the inclusion $\mathcal{Q}$-functor $j\colon (E, \sigma) \hookrightarrow (X, \alpha)$, we have a left adjoint $\mathcal{Q}$-distributor

$$j_\natural \circ \mu\colon \{q\} \to\!\!\!\!\!\circ\!\!\to (X, \alpha).$$

Since (X, α) is Cauchy complete, there exists $x \in X$ such that $j_\natural \circ \mu = \alpha(x, -)$. It follows that

$$\beta(fx, -) = f_\natural \circ \alpha(x, -) = f_\natural \circ j_\natural \circ \mu = g_\natural \circ j_\natural \circ \mu = g_\natural \circ \alpha(x, -) = \beta(gx, -),$$

where the third equality follows from $fj = gj$. Thus, the separatedness of (Y, β) forces $fx = gx$, and consequently $x \in E$. Therefore,

$$\mu = \sigma \circ \mu = j_\natural^* \circ j_\natural \circ \mu = j_\natural^* \circ \alpha(x, -) = \alpha(x, j-) = \sigma(x, -),$$

where the second equality follows from Lemma 3.2, as desired. □

4. Quantale-valued sets as enriched categories

We are now ready to introduce the key notion of this paper:

Definition 4.1. (See [10, 12, 15, 23].) A Q-*set* a set X equipped with a map

$$\alpha\colon X\times X\longrightarrow \mathsf{Q},$$

such that

(S1) $\alpha(x,y)\leqslant\alpha(x,x)\wedge\alpha(y,y)$,

(S2) $\alpha(x,y)=\alpha(y,x)$,

(S3) $\alpha(y,z)\ \&\ (\alpha(y,y)\rightarrow\alpha(x,y))\leqslant\alpha(x,z)$

for all $x,y,z\in X$.

In order to exhibit Q-sets as enriched categories, we need the following quantaloid constructed from Q:

Proposition 4.2. *(See [15, 23, 28].) The following data define an involutive quantaloid* DQ*:*

- ob $\mathsf{DQ}=\mathsf{Q}$;
- $\mathsf{DQ}(p,q)=\{u\in\mathsf{Q}\mid u\leqslant p\wedge q\}$;
- *the composite of $u\in\mathsf{DQ}(p,q)$ and $v\in\mathsf{DQ}(q,r)$ is given by $v\circ_q u$ (see (2.i));*
- *the identity* DQ*-arrow on $q\in\mathsf{Q}$ is q itself;*
- *each hom-set $\mathsf{DQ}(p,q)$ is equipped with the order inherited from* Q*;*
- *the involution of $u\colon p\longrightarrow q$ in* DQ *is given by $u\colon q\longrightarrow p$.*

From the definition we see that a DQ-category consists of a set X, a map $|\text{-}|\colon X\longrightarrow\mathsf{Q}$ and a map $\alpha\colon X\times X\longrightarrow\mathsf{Q}$ such that

$$\alpha(x,y)\leqslant|x|\wedge|y|,\quad |x|\leqslant\alpha(x,x)\quad\text{and}\quad\alpha(y,z)\ \&\ (|y|\rightarrow\alpha(x,y))\leqslant\alpha(x,z)$$

for all $x,y,z\in X$. Note that the first and the second inequalities above force

$$\alpha(x,x)=|x| \tag{4.i}$$

for all $x\in X$. Thus, a DQ-category is exactly given by a map $\alpha\colon X\times X\longrightarrow\mathsf{Q}$ satisfying (S1) and (S3) of Definition 4.1, and a Q-set is precisely a DQ-category satisfying (S2). Therefore:

Proposition 4.3. *(See [15, Proposition 6.3].) A* Q*-set is precisely a symmetric* DQ*-category.*

We denote by

$$\mathsf{Q}\text{-}\mathbf{Set}$$

the category whose objects are Q-sets, and whose morphisms are left adjoint DQ-distributors.

Remark 4.4. If $\mathsf{Q}=\Omega$ is a frame, then we obtain the well-known category Ω-**Set** as considered in [5] and [3, Sections 2.8 and 2.9]. Since Ω-**Set** is equivalent to the category $\mathbf{Sh}(\Omega)$ of sheaves on Ω (cf. [5, Theorem 5.9] and [3, Theorem 2.9.8]), it follows that Ω-**Set** is a topos (see [5, Proposition 9.2] and and [3, Example 5.2.3]).

Given a Q-set (X,α), a left adjoint DQ-distributor $\mu\colon\{q\}\longrightarrow\!\!\!\!\!\circ\ (X,\alpha)$ is usually called a *singleton* [15, 23], and it can be characterized as follows:

Proposition 4.5. *(See [15, Definition 6.4] and [23, Definition 5.3].) A singleton on a* Q*-set (X,α) is precisely a map $\mu\colon X\longrightarrow\mathsf{Q}$ such that* (ss1) $\mu(x) \leqslant \alpha(x,x)$,

(ss2) $\mu(x) \,\&\, (\alpha(x,x) \to \alpha(x,y)) \leqslant \mu(y)$,

(ss3) $|\mu| \leqslant \bigvee_{x\in X} \mu(x) \,\&\, (\alpha(x,x) \to \mu(x))$,

(ss4) $\mu(x) \,\&\, (|\mu| \to \mu(y)) \leqslant \alpha(x,y)$

for all $x, y \in X$, where $|\mu| = \bigvee_{x\in X} \mu(x)$. In this case, it necessarily holds that $\mu^*(x) = \mu(x)$ for all $x \in X$.

Note that Q-**Set** and DQ-**CcSymCat** are exactly the categories **S**M-**Mod**$_{\rm ad}$ and **C**M-**Set** in [23], respectively. Hence:

Proposition 4.6. *(See [23, Proposition 5.7(2)].) The category* Q-**Set** *is equivalent to the full subcategory*

$$\mathsf{DQ}\text{-}\mathbf{CcSymCat}$$

of DQ-**Cat** *whose objects are separated and Cauchy complete* Q*-set.*

From the construction of DQ (Proposition 4.2) and [26, Remark 2.4(1)] we see that the terminal object of the category DQ-**Cat** is given by

$$(\mathsf{Q}, \wedge),$$

whose underlying set is Q equipped with the identity map as its type map, and whose hom-arrows are given by $p \wedge q$ for all $p, q \in \mathsf{Q}$. The following proposition tells us that $(\mathsf{Q}, \wedge)$ is also the terminal object of DQ-**CcSymCat**:

Proposition 4.7. *(See [23, Example 4.8].)* $(\mathsf{Q}, \wedge)$ *is a separated and Cauchy complete* Q*-set.*

The following proposition guarantees that the Cauchy completion of a Q-set is also a Q-set:

Proposition 4.8. *(See [23, Proposition 5.4].) The Cauchy completion $(\widehat{X}, \widehat{\alpha})$ of a symmetric* DQ*-category (X, α) is also symmetric.*

By Proposition 4.5 it is straightforward to check that

$$p \sqsubseteq q \iff p \text{ is a singleton on the one-element } \mathsf{Q}\text{-set } \{q\}. \tag{4.ii}$$

Thus, the following characterization of the set C_q (see (2.iii)) can be deduced from (3.vii) in combination with Propositions 2.5 and 4.2:

Proposition 4.9. *(See [23, Example 4.13].) For each $q \in \mathsf{Q}$, the Cauchy completion of the one-element* Q*-set $\{q\}$ is precisely C_q equipped with the* Q*-set structure inherited from* $(\mathsf{Q}, \wedge)$*.*

For $p, q \in \mathsf{Q}$, since a DQ-functor must preserve types, a DQ-functor from $(C_p, \wedge)$ to $(C_q, \wedge)$, whenever it exists, is necessarily neutral on every element of C_p. Hence, there is at most one DQ-functor from $(C_p, \wedge)$ to $(C_q, \wedge)$, which can only be the inclusion map and exists only when $C_p \subseteq C_q$. Correspondingly, the order $\sqsubseteq$ may be described as follows:

Proposition 4.10. *Let $p, q \in \mathsf{Q}$. Then $p \sqsubseteq q$ if, and only if, there exists a (unique)* DQ*-functor from $(C_p, \wedge)$ to $(C_q, \wedge)$.*

Proof. If $p \sqsubseteq q$, then by Proposition 2.6, every $r \in C_p$ belongs to C_q; that is, $C_p \subseteq C_q$. Thus, the inclusion map is the unique DQ-functor from $(C_p, \wedge)$ to $(C_q, \wedge)$. Conversely, if the inclusion map is a DQ-functor from $(C_p, \wedge)$ to $(C_q, \wedge)$, then $p \in C_q$, which means that $p \sqsubseteq q$. □

The following proposition reveals that in a separated and Cauchy complete Q-set (X, α), an element $x \in X$ has a unique "restriction" on each $p \sqsubseteq \alpha(x,x)$:

Proposition 4.11. *Let (X,α) be a separated and Cauchy complete Q-set. For each $x \in X$ with $\alpha(x,x) = q$, if $p \sqsubseteq q$, then there exists a unique element $x_p \in X$ such that*

$$\alpha(x_p, x_p) = p \quad \textit{and} \quad \alpha(x_p, y) = \alpha(x,y) \wedge p \tag{4.iii}$$

for all $y \in X$.

Proof. For each $y \in Y$, since $\alpha(x,y) \leqslant \alpha(x,x) = q$ and p is idempotent in the quantale $({\downarrow}\, q, \circ_q)$ (see Proposition 2.4 and (2.ii)), we have

$$\alpha(x,y) \wedge p = \alpha(x,y) \circ_q p.$$

Thus, $\alpha(x,-) \wedge p$ is the composite of singletons

$$\{p\} \overset{p}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;\;} \{q\} \overset{\alpha(x,-)}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;\;} (X,\alpha),$$

which is again a singleton on (X,α). Hence, the existence and uniqueness of $x_p \in X$ satisfying (4.iii) follow from the Cauchy completeness and separatedness of (X,α). $\square$

Proposition 4.12. *Let $q \in \mathsf{Q}$, and let (Y,β) be a separated and Cauchy complete Q-set. Then every DQ-functor from $(C_q, \wedge)$ to (Y,β) is uniquely determined by an element $y \in Y$ with $\beta(y,y) = q$, which is exactly*

$$\overline{y}\colon (C_q, \wedge) \longrightarrow (Y,\beta), \quad p \mapsto y_p. \tag{4.iv}$$

Proof. Since a DQ-functor from $\{q\}$ to (Y,β) is essentially an element $y \in Y$ with $\beta(y,y) = q$, the conclusion is an immediate consequence of (3.xi) and (3.xii) together with Propositions 4.9 and 4.11. $\square$

Proposition 4.13. *Let $q \in \mathsf{Q}$, and let (Y,β) be a Q-set. Then every $y \in Y$ with $\beta(y,y) = q$ induces a DQ-functor*

$$\widehat{y}\colon (C_q, \wedge) \longrightarrow (\widehat{Y}, \widehat{\beta}), \quad p \mapsto \beta(y,-) \wedge p. \tag{4.v}$$

Proof. Note that $\widehat{y}$ is obtained by applying (3.x) to the DQ-functor

$$y\colon \{q\} \longrightarrow (Y,\beta)$$

targeting at y. Indeed, by (3.x) we have

$$(\widehat{y}p)(y') = y_\natural(y') \circ_q p = \beta(y,y') \circ_q p = \beta(y,y') \wedge p$$

for all $p \in C_q$, $y' \in Y$, where the last equality holds because $\beta(y,y') \leqslant \beta(y,y) = q$ and p is idempotent in the quantale $({\downarrow}\, q, \circ_q)$ (see Proposition 2.4). $\square$

5. Q-Set is a topos only if Q is a frame

Let us recall the following facts in a topos:

Proposition 5.1. *(see [20, Proposition IV.1.2].) Every monic arrow in a topos is an equalizer.*

Proposition 5.2. *(see [20, Proposition IV.6.3] and [3, Proposition 5.10.2].) Let $\mathcal{E}$ be a topos. For each $X \in \operatorname{ob}\mathcal{E}$, the partially ordered set $\operatorname{Sub} X$ is a lattice, in which the intersection $U \cap V$ of subobjects U, V of X is given by the pullback on the left below, and their union $U \cup V$ is given by the pushout on the right below.*

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \rightarrowtail & X \end{array} \qquad\qquad \begin{array}{ccc} U \cap V & \rightarrowtail & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

Lemma 5.3. *Suppose that* DQ-**CcSymCat** *is a topos. Then for each $q \in \mathsf{Q}$, every subobject of $(C_q, \wedge)$ in* DQ-**CcSymCat** *is of the form $(C_p, \wedge)$ for some $p \sqsubseteq q$.*

Proof. By Propositions 3.3 and 5.1, we may assume that a subobject of $(C_q, \wedge)$ in DQ-**CcSymCat** is a subset $S \subseteq C_q$ equipped with the inherited DQ-category structure $\wedge$. We show that $S = C_p$ for some $p \sqsubseteq q$.

First, S is a $\sqsubseteq$-lower set. Suppose that $r \in S$ and $r' \sqsubseteq r$. Then

$$\mu\colon S \longrightarrow \mathsf{Q}, \quad \mu(s) = s \wedge r'$$

is a singleton on $(S, \wedge)$. Indeed, it is straightforward to verify that μ satisfies (ss1), (ss2) and (ss4) of Proposition 4.5. For (ss3), from $r' \sqsubseteq r$ and (2.iv) we see that

$$|\mu| = r' = r' \,\&\, (r \to r') = (r \wedge r') \,\&\, (r \to (r \wedge r')) \leqslant \bigvee_{s \in S} (s \wedge r') \,\&\, (s \to (s \wedge r')).$$

Since $(S, \wedge)$ lies in DQ-**CcSymCat**, it is Cauchy complete, and thus there exists $s_0 \in S$ such that $\mu(s) = s \wedge s_0$, i.e.,

$$s \wedge r' = s \wedge s_0, \tag{5.i}$$

for all $s \in S$. In particular, setting $s = r$ and $s = s_0$ respectively in (5.i) we have

$$r' = r \wedge r' = r \wedge s_0 \quad \text{and} \quad s_0 \wedge r' = s_0,$$

which imply that $r' \leqslant s_0$ and $s_0 \leqslant r'$, respectively. Hence $r' = s_0 \in S$.

Second, S has a $\sqsubseteq$-maximum element. By checking the conditions of Proposition 4.5 we may also see that

$$\lambda\colon S \longrightarrow \mathsf{Q}, \quad \lambda(s) = s$$

is a singleton on $(S, \wedge)$. Thus, the Cauchy completeness of $(S, \wedge)$ guarantees the existence of $p \in S$ such that $\lambda(s) = s \wedge p$, i.e., $s = s \wedge p$ for all $s \in S$. Thus $s \leqslant p$ for all $s \in S$; that is, p is the $\leqslant$-maximum element of S, which is also the $\sqsubseteq$-maximum element of S (see Proposition 2.10). □

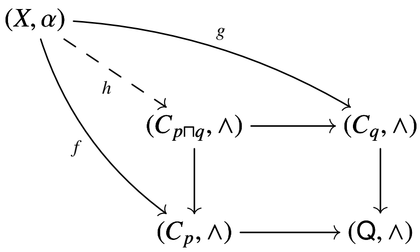
Lemma 5.4. *Suppose that* DQ-**CcSymCat** *is a topos. Then*

$$p \sqcap q = p \wedge q$$

for all $p, q \in \mathsf{Q}$.

Proof. By Propositions 3.3, 4.9 and 5.1, $(C_p, \wedge)$ and $(C_q, \wedge)$ are subobjects of $(\mathsf{Q}, \wedge)$ in DQ-**CcSymCat**. Our strategy is to explore their union in $\mathrm{Sub}(\mathsf{Q}, \wedge)$ under the guidance of Proposition 5.2.

Step 1. The intersection of $(C_p, \wedge)$ and $(C_q, \wedge)$ in $\mathrm{Sub}(\mathsf{Q}, \wedge)$ is $(C_{p \sqcap q}, \wedge)$. It suffices to show that the inner square of the diagram



(5.ii)

is a pullback. In fact, the commutativity of the inner square of (5.ii) is an immediate consequence of Propositions 2.9 and 4.10, in which every arrow is the inclusion DQ-functor. Moreover, if the outer quadrilateral of (5.ii) is commutative, then

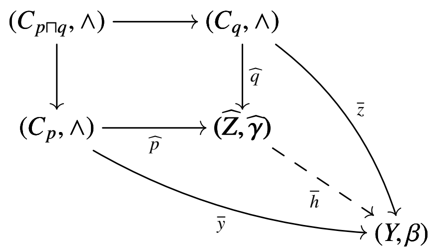
$$hx := fx = gx \in C_p \cap C_q$$

for all $x \in X$. Thus, it follows from (2.iv) and Proposition 2.9 that $hx \sqsubseteq p \sqcap q$, i.e., $hx \in C_{p\sqcap q}$. Therefore, h is the unique map making the triangles in (5.ii) commutative, and its DQ-functoriality follows immediately from that of f and g.

Step 2. The union of $(C_p, \wedge)$ and $(C_q, \wedge)$ in $\mathrm{Sub}(\mathsf{Q}, \wedge)$ is given by the Cauchy completion of the Q-set (Z, γ) with

$$Z = \{p, q\} \quad \text{and} \quad \gamma(p, p) = p, \quad \gamma(q, q) = q, \quad \gamma(p, q) = \gamma(q, p) = p \sqcap q.$$

To this end, it suffices to show that the inner square of the diagram



(5.iii)

is a pushout, where $\widehat{p}$ and $\widehat{q}$ are the DQ-functors defined by applying (3.x) to the inclusion DQ-functors $\{p\} \hookrightarrow (Z, \gamma)$ and $\{q\} \hookrightarrow (Z, \gamma)$ targeting at p and q, respectively.

First, the inner square of (5.iii) is commutative. Suppose that $r \in C_{p\sqcap q}$. Then $r \sqsubseteq p \sqcap q \sqsubseteq p$. Thus, it follows from Proposition 4.13 that

$$(\widehat{p}r)(p) = \gamma(p, p) \wedge r = p \wedge r = r = (p \sqcap q) \wedge r = \gamma(p, q) \wedge r = (\widehat{p}r)(q).$$

Similarly, $(\widehat{q}r)(p) = (\widehat{q}r)(q) = r$. Hence $\widehat{p}r = \widehat{q}r$, as desired.

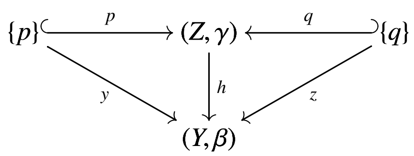
Second, suppose that the outer quadrilateral of (5.iii) is commutative, where $\overline{y}$ and $\overline{z}$ are the DQ-functors determined by $y \in Y$ with $\beta(y, y) = p$ and $z \in Y$ with $\beta(z, z) = q$, respectively (see Proposition 4.12). Then, Proposition 4.12 implies that

$$y_r = \overline{y}r = \overline{z}r = z_r$$

for all $r \in C_{p\sqcap q}$. We show that the transpose $\overline{h}\colon (\widehat{Z}, \widehat{\gamma}) \longrightarrow (Y, \beta)$ (see (3.xi) and (3.xii)) of

$$h\colon (Z, \gamma) \longrightarrow (Y, \beta), \quad hp = y \quad \text{and} \quad hq = z$$

is the unique DQ-functor such that the two triangles in (5.iii) are commutative. Indeed, by applying the functor $\widehat{(-)}$ to the commutative triangles



we obtain that

$$\widehat{h}\widehat{p} = \widehat{y} \quad \text{and} \quad \widehat{h}\widehat{q} = \widehat{z}.$$

Then, by the definition of $\overline{(-)}$ (see (3.xii)) we conclude that

$$\overline{h}\widehat{p} = \overline{y} \quad \text{and} \quad \overline{h}\widehat{q} = \overline{z}.$$

For the uniqueness of $\overline{h}$, suppose that $\overline{k}\colon (\widehat{Z}, \widehat{\gamma}) \longrightarrow (Y, \beta)$ is the transpose of $k\colon (Z, \gamma) \longrightarrow (Y, \beta)$ (see (3.xi) and (3.xii)) and satisfies $\overline{k}\widehat{p} = \overline{y}$ and $\overline{k}\widehat{q} = \overline{z}$. Then it follows soon from the adjoint property (i.e., the natural bijection (3.xi)) that $kp = y$ and $kp = z$. Hence $k = h$.

Step 3. Since $(\widehat{Z}, \widehat{\gamma})$ is the union of $(C_p, \wedge)$ and $(C_q, \wedge)$ in $\mathrm{Sub}(\mathsf{Q}, \wedge)$, it is also a subobject of $(\mathsf{Q}, \wedge)$. As (Z, γ) is a clearly subobject of $(\widehat{Z}, \widehat{\gamma})$ (see (3.viii)), we conclude that (Z, γ) is a subobject of $(\mathsf{Q}, \wedge)$. Therefore, it follows from Propositions 3.3 and 5.1 that

$$p \sqcap q = \gamma(p, q) = p \wedge q,$$

which completes the proof. □

Proposition 5.5. DQ-**CcSymCat** *is a topos if, and only if,* Q *is a frame.*

Proof. The "if" part is a direct consequence of Remark 4.4 and Proposition 4.6. For the "only if" part, suppose that DQ-**CcSymCat** is a topos. Setting $p = 1$ in Lemma 5.4 we obtain that $q \sqcap 1 = q \wedge 1 = q$, and consequently $q \sqsubseteq 1$ for all $q \in \mathsf{Q}$. By (2.iv), this means that

$$q \,\&\, q = q \,\&\, (1 \to q) = q$$

for all $q \in \mathsf{Q}$; that is, every element of Q is idempotent. Hence Q is a frame. □

Therefore, the main result of this paper is an immediate consequence of Propositions 4.6 and 5.5:

Theorem 5.6. Q-**Set** *is a topos if, and only if,* Q *is a frame.*

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