# Quantale-valued maps and partial maps

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**Abstract**

Let Q be a commutative and unital quantale. By a Q-map we mean a left adjoint in the quantaloid of sets and Q-relations, and by a partial Q-map we refer to a Kleisli morphism with respect to the maybe monad on the category Q-**Map** of sets and Q-maps. It is shown that every Q-map is symmetric if and only if Q is weakly lean, and that every Q-map is exactly a map in **Set** if and only Q is lean. Moreover, assuming the axiom of choice, it is shown that the category of sets and partial Q-maps is monadic over Q-**Map**.

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## 1. Introduction

Given (crisp) sets $X$ and $Y$, a *partial map* $f$ from $X$ to $Y$ is a map from a (possibly empty) subset $X' \subseteq X$ to $Y$. For the category

$$\mathbf{Set}^{\partial}$$

of (crisp) sets and partial maps, the following results are well known [17, 21]:

- $\mathbf{Set}^{\partial}$ is equivalent to the coslice category $\{\star\}/\mathbf{Set}$, where $\{\star\}$ is the singleton set.
- The forgetful functor $\mathsf{U}\colon \{\star\}/\mathbf{Set} \longrightarrow \mathbf{Set}$ admits a left adjoint, which carries a set $X$ to the inclusion map

  $$\{\star\} \hookrightarrow X \amalg \{\star\},$$

  where $X \amalg \{\star\}$ refers to the disjoint union of $X$ and the singleton set.
- The induced monad on **Set** is called the *maybe monad*, whose Eilenberg-Moore and Kleisli categories are $\{\star\}/\mathbf{Set}$ and $\mathbf{Set}^{\partial}$, respectively (cf. [21, Examples 5.1.4 and 5.2.10]).

Therefore, the Eilenberg-Moore and Kleisli categories of the maybe monad on **Set** are equivalent. In particular, $\{\star\}/\mathbf{Set}$ and $\mathbf{Set}^{\partial}$ are both monadic over **Set**.

It is then natural to ask whether it is possible to formulate the notion of partial map, and establish the above results in the fuzzy setting. To achieve this goal, we note that there is a very general framework in category theory for partial maps; see, e.g., [17, Exercise IV.2.11]. Explicitly, let $\mathcal{C}$ be a category with finite coproducts. For every $\mathcal{C}$-object $A$, the forgetful functor from the coslice category $A/\mathcal{C}$ to $\mathcal{C}$, i.e.,

$$\mathsf{U}\colon A/\mathcal{C} \longrightarrow \mathcal{C}, \quad (A \to C) \mapsto C, \tag{1.i}$$

admits a left adjoint, given by

$$C \mapsto (A \to A \amalg C). \tag{1.ii}$$

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It is not difficult to see that the Eilenberg-Moore category of the induced monad is isomorphic to $A/\mathcal{C}$, and thus $A/\mathcal{C}$ is monadic over $\mathcal{C}$. The maybe monad on **Set** is simply obtained by setting $\mathcal{C} = \mathbf{Set}$ and $A = \{\star\}$.

Considering a non-trivial, commutative and unital *quantale* [22] $\mathsf{Q}$ as the table of truth values and following the terminology of [10, Definition 2.3.1], by a $\mathsf{Q}$-*map* (i.e., a map valued in the quantale $\mathsf{Q}$) we mean a left adjoint in the *quantaloid* [23, 26]

$$\mathsf{Q}\text{-}\mathbf{Rel}$$

of sets and $\mathsf{Q}$-relations; that is, a $\mathsf{Q}$-relation $\zeta\colon X \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\; Y$ admitting a right adjoint in $\mathsf{Q}$-**Rel**, with the value

$$\zeta(x, y)$$

interpreted as the extent of $y$ being the image of $x$ under the map $\zeta$ (see Remark 3.2 for detailed elaboration). In Section 3 we prove the following results (Theorems 3.12 and 3.15), which reveal that for a general quantale $\mathsf{Q}$, $\mathsf{Q}$-maps are essentially different from crisp maps between sets:

- Every $\mathsf{Q}$-map is *symmetric* (in the sense that the right adjoint is given by its opposite) if, and only if, $\mathsf{Q}$ is *weakly lean*. This is a discrete counterpart of [10, Proposition 3.5.3].

- Every $\mathsf{Q}$-map is the graph of a map in **Set** (so that the category $\mathsf{Q}$-**Map** of sets and $\mathsf{Q}$-maps is isomorphic to **Set**) if, and only if, $\mathsf{Q}$ is *lean*. This is a generalization of [12, Proposition III.1.2.1] to the non-integral setting.

With the above preparations we are now able to postulate the notion of *partial* $\mathsf{Q}$-*map* through the *maybe monad* on the category $\mathsf{Q}$-**Map**, which is obtained by setting $\mathcal{C} = \mathsf{Q}\text{-}\mathbf{Map}$ and $A = \{\star\}$ in (1.i) and (1.ii). Explicitly, by a partial $\mathsf{Q}$-map $\zeta$ from $X$ to $Y$ we refer to a $\mathsf{Q}$-map

$$\zeta\colon X \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\; Y \amalg \{\star\},$$

which is precisely a morphism in the Kleisli category of the maybe monad on $\mathsf{Q}$-**Map**, with the value

$$\zeta(x, \star)$$

interpreted as the degree that $x$ has no image under the map $\zeta$ (see Remark 4.7 for detailed elaboration). The main result of Section 4, Theorem 4.10, is to show that assuming the axiom of choice, the Eilenberg-Moore category (i.e., the coslice category $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$) and the Kleisli category (i.e., the category $\mathsf{Q}\text{-}\mathbf{Map}^{\partial}$ of sets and partial $\mathsf{Q}$-maps) of the maybe monad on $\mathsf{Q}$-**Map** are equivalent, so that $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$ and $\mathsf{Q}\text{-}\mathbf{Map}^{\partial}$ are both monadic over $\mathsf{Q}$-**Map**. It is noteworthy to point out that this result may not hold in the general categorical setting (i.e., the monad induced by (1.i) and (1.ii) for a general category $\mathcal{C}$), and an equivalent statement of the axiom of choice given in [9] (i.e., every non-empty set admits a group structure) plays a vital role in its proof.

## 2. Quantales

Throughout, let

$$\mathsf{Q} = (\mathsf{Q}, \&, k)$$

denote a non-trivial, commutative and unital *quantale* [18, 22, 7], which consists of a complete lattice $\mathsf{Q}$ (with the top and bottom elements denoted by $\top$ and $\bot$, respectively), an element $k \in \mathsf{Q}$ and a binary operation $\&$ on $\mathsf{Q}$, such that

- $\bot < k$,
- $(\mathsf{Q}, \&, k)$ is a commutative monoid with unit $k$, and
- $p \;\&\; \Big(\bigvee\limits_{i\in I} q_i\Big) = \bigvee\limits_{i\in I} p \;\&\; q_i$ for all $p, q_i \in \mathsf{Q}\,(i \in I)$.

The right adjoint induced by the multiplication &, denoted by $\to$,

$$(p \mathbin{\&} -) \dashv (p \to -) \colon \mathsf{Q} \longrightarrow \mathsf{Q},$$

satisfies

$$p \mathbin{\&} q \leqslant r \iff p \leqslant q \to r$$

for all $p, q, r \in \mathsf{Q}$.

We say that $\mathsf{Q}$ is *integral* if $k = \top$, and $\mathsf{Q}$ is *divisible* [13, 8] if

$$d \leqslant q \implies \exists p \in \mathsf{Q} \colon d = p \mathbin{\&} q$$

for all $d, q \in \mathsf{Q}$. In particular, divisible quantales are necessarily integral (see, e.g., [20, Proposition 2.1]).

**Example 2.1.** We list here some examples of commutative and unital quantales that will be concerned later:

(1) On the three-chain $C_3 = \{\bot, k, \top\}$ we have the quantale $(C_3, \&, k)$, with

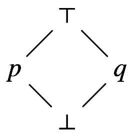
$$\top \mathbin{\&} \top = \top$$

and the other multiplications being trivial.

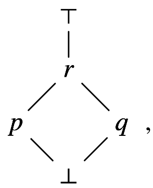
(2) Every frame is a divisible quantale. In particular, the partially ordered sets

$$F_1 = \{\bot, p, q, \top\} \quad \text{and} \quad F_2 = \{\bot, p, q, r, \top\}$$

illustrated by the Hasse diagrams

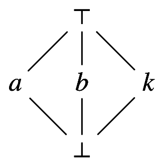



and




respectively, are both frames.

(3) On the diamond lattice $M_3$ given by the Hasse diagram




we have the quantale $(M_3, \&, k)$, with

$$a \mathbin{\&} a = b, \quad b \mathbin{\&} b = a, \quad a \mathbin{\&} b = k \quad \text{and} \quad a \mathbin{\&} \top = b \mathbin{\&} \top = \top.$$

(4) Every continuous t-norm [14, 15, 1] $*$ on the unital interval $[0, 1]$ gives rise to a divisible quantale $([0, 1], *, 1)$.

(5) Let $[-\infty, \infty]$ be the extended real line equipped with the order "$\geqslant$". Then $([-\infty, \infty], +, 0)$ is a quantale, which includes the Lawvere quantale [16] $([0, \infty], +, 0)$ as a divisible subquantale.

(6) Every commutative monoid $(M, \&, k)$ induces a *free quantale* $(\mathsf{P}M, \&, \{k\})$, where $\mathsf{P}M$ is the powerset of $M$, and

$$A \mathbin{\&} B = \{a \mathbin{\&} b \mid a \in A,\ b \in B\}$$

for all $A, B \subseteq M$.

A *quantaloid* [23, 25, 26] $\mathcal{Q}$ is a category with a class of objects $\mathcal{Q}_0$ in which $\mathcal{Q}(p,q)$ is a complete lattice for all $p,q\in\mathcal{Q}_0$, such that the composition $\circ$ of morphisms preserves joins on both sides, i.e.,

$$v\circ\Big(\bigvee_{i\in I}u_i\Big)=\bigvee_{i\in I}(v\circ u_i)\quad\text{and}\quad\Big(\bigvee_{i\in I}v_i\Big)\circ u=\bigvee_{i\in I}(v_i\circ u)$$

for all $u,u_i\in\mathcal{Q}(p,q)$, $v,v_i\in\mathcal{Q}(q,r)$ $(i\in I)$. The corresponding right adjoints induced by the compositions

$$(-\circ u)\dashv(-\swarrow u)\colon\mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(q,r)\quad\text{and}\quad(v\circ -)\dashv(v\searrow -)\colon\mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(p,q)$$

satisfy

$$v\circ u\leqslant w\iff v\leqslant w\swarrow u\iff u\leqslant v\searrow w$$

for all $\mathcal{Q}$-arrows $u\colon p\longrightarrow q$, $v\colon q\longrightarrow r$, $w\colon p\longrightarrow r$.

An *adjunction* in a quantaloid $\mathcal{Q}$ is a pair of $\mathcal{Q}$-arrows $u\colon p\longrightarrow q$ and $v\colon q\longrightarrow p$, denoted by $u\dashv v$, such that

$$1_p\leqslant v\circ u\quad\text{and}\quad u\circ v\leqslant 1_q.$$

**Proposition 2.2.** *If $u\dashv v\colon q\longrightarrow p$ in a quantaloid $\mathcal{Q}$, then*

$$v=u\searrow 1_q\quad\textit{and}\quad u=1_q\swarrow v.$$

*Proof.* Note that $1_p\leqslant v\circ u$ implies that $u\searrow 1_q\leqslant v\circ u\circ(u\searrow 1_q)\leqslant v$, and the reverse inequality is an immediate consequence of $u\circ v\leqslant 1_q$. Therefore $v=u\searrow 1_q$, and $u=1_q\swarrow v$ can be proved analogously. □

Therefore, the right adjoint of a $\mathcal{Q}$-arrow, when it exists, is necessarily unique. In what follows we define

$$u^*:=u\searrow 1_q\colon q\longrightarrow p\tag{2.i}$$

for each $\mathcal{Q}$-arrow $u\colon p\longrightarrow q$. $u$ is called a *map* in $\mathcal{Q}$ [10] if $u\dashv u^*$, and we denote by

$$\mathbf{Map}(\mathcal{Q})$$

the subcategory of $\mathcal{Q}$ whose objects are the same as $\mathcal{Q}$, and whose morphisms are maps in $\mathcal{Q}$.

Since it always holds that $u\circ u^*=u\circ(u\searrow 1_q)\leqslant 1_q$, we immediately obtain the following lemma:

**Lemma 2.3.** *A $\mathcal{Q}$-arrow $u\colon p\longrightarrow q$ is a map in $\mathcal{Q}$ if, and only if, $1_p\leqslant u^*\circ u$.*

Given sets $X$, $Y$, a $\mathsf{Q}$-*relation* $\varphi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a function $\varphi\colon X\times Y\longrightarrow\mathsf{Q}$. We denote by

$$\varphi^{\mathrm{op}}:Y\relbar\joinrel\mapstochar\joinrel\rightarrow X,\quad\varphi^{\mathrm{op}}(y,x)=\varphi(x,y)$$

the *opposite* of a $\mathsf{Q}$-relation $\varphi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Y$. Sets and $\mathsf{Q}$-relations constitute a quantaloid

$$\mathsf{Q}\text{-}\mathbf{Rel},$$

in which the local order is inherited from $\mathsf{Q}$, and

$$\psi\circ\varphi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Z,\quad(\psi\circ\varphi)(x,z)=\bigvee_{y\in Y}\psi(y,z)\,\&\,\psi(x,y),\tag{2.ii}$$

$$\xi\swarrow\varphi\colon Y\relbar\joinrel\mapstochar\joinrel\rightarrow Z,\quad(\xi\swarrow\varphi)(y,z)=\bigwedge_{x\in X}(\varphi(x,y)\rightarrow\xi(x,z)),\tag{2.iii}$$

$$\psi\searrow\xi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Y,\quad(\psi\searrow\xi)(x,y)=\bigwedge_{z\in Z}\psi(y,z)\rightarrow\xi(x,z)\tag{2.iv}$$

for all $\mathsf{Q}$-relations $\varphi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $\psi\colon Y\relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $\xi\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow Z$; the identity $\mathsf{Q}$-relation on $X$ is given by

$$\mathrm{id}\colon X\relbar\joinrel\mapstochar\joinrel\rightarrow X,\quad\mathrm{id}_X(x,y)=\begin{cases}k & \text{if } x=y,\\ \bot & \text{else.}\end{cases}$$

Given Q-relations $\varphi\colon X \rightarrow\!\!\!\!\!+ Y$ and $\psi\colon Z \rightarrow\!\!\!\!\!+ W$, their disjoint union

$$\psi \amalg \varphi\colon X \amalg Z \rightarrow\!\!\!\!\!+ Y \amalg W$$

is given by

$$(\psi \amalg \varphi)(x, y) = \varphi(x, y), \quad (\psi \amalg \varphi)(z, w) = \psi(z, w) \quad \text{and} \quad (\psi \amalg \varphi)(x, w) = (\psi \amalg \varphi)(z, y) = \bot$$

for all $x \in X$, $z \in Z$, $y \in Y$, $w \in W$. It is straightforward to verify that

$$(\psi \circ \varphi) \amalg \xi = (\psi \amalg \xi) \circ (\varphi \amalg \xi) \tag{2.v}$$

for all Q-relations $\varphi\colon X \rightarrow\!\!\!\!\!+ Y$, $\psi\colon Y \rightarrow\!\!\!\!\!+ Z$, $\xi\colon Z \rightarrow\!\!\!\!\!+ W$.

## 3. Quantale-valued maps

**Definition 3.1.** Let $X$, $Y$ be (crisp) sets. A Q-*map* $\zeta$ from $X$ to $Y$ is a map $\zeta\colon X \rightarrow\!\!\!\!\!+ Y$ in the quantaloid Q-**Rel**.

Sets and Q-maps constitute a category

$$\text{Q-}\mathbf{Map} := \mathbf{Map}(\text{Q-}\mathbf{Rel}).$$

**Remark 3.2.** To justify Definition 3.1, let $\zeta\colon X \rightarrow\!\!\!\!\!+ Y$ be a Q-map. First of all, the value

$$\zeta(x, y)$$

is interpreted as the extent of $y$ being the image of $x$ under the map $\zeta$. By Equations (2.i) and (2.iv), the value

$$\zeta^*(y, x) = (\zeta \searrow \mathrm{id}_Y)(y, x) = \bigwedge_{z \in Y} \zeta(x, z) \to \mathrm{id}_Y(y, z) \tag{3.i}$$

also represents the extent of $y$ being the image of $x$ under the map $\zeta$, since the last expression of (3.i) may be understood as:

- For each $z \in Y$, if $z$ is the image of $x$ under $\zeta$, then $z$ is equal to $y$.

Therefore, the adjunction $\zeta \dashv \zeta^*$ can be translated as follows:

- For every $x \in X$, there exists $y \in Y$ such that $y$ is the image of $x$ under $\zeta$; because $\mathrm{id}_X \leqslant \zeta^* \circ \zeta$ means that

$$k \leqslant \bigvee_{y \in Y} \zeta^*(y, x) \,\&\, \zeta(x, y) \tag{3.ii}$$

  for all $x \in X$.

- If $y, z \in Y$ are both the images of $x$ under $\zeta$, then $y$ is equal to $z$; because $\zeta \circ \zeta^* \leqslant \mathrm{id}_Y$ means that

$$\bigvee_{x \in X} \zeta(x, z) \,\&\, \zeta^*(y, x) \leqslant \mathrm{id}_Y(y, z) \tag{3.iii}$$

  for all $y, z \in Y$.

**Remark 3.3.** The notion of Q-map should be carefully distinguished from related notions in the literature:

- *Fuzzy mappings* in [2] and *fuzzifying functions* in [6] are precisely our Q-relations.
- When Q is an integral quantale, every Q-map is a *fuzzy function* with respect to $\mathrm{id}_X$ and $\mathrm{id}_Y$ in the sense of [24, 3] or [4, 5], but not vice versa.

- When Q is a frame, Q-maps are exactly Q-*valued fuzzy functions* in the sense of [19].

**Lemma 3.4.** *Let $\zeta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ be a* Q*-map. Then*

$$(\zeta^* \circ \zeta)(x,x) = k \quad \textit{and} \quad \zeta(x,z) \mathbin{\&} \zeta^*(y,x) = \bot$$

*for all $x \in X$, $y,z \in Y$ with $y \neq z$.*

*Proof.* The first equality follows from

$$\begin{aligned} k = \mathrm{id}_X(x,x) &\leqslant (\zeta^* \circ \zeta)(x,x) = \bigvee_{y\in Y} \zeta^*(y,x) \mathbin{\&} \zeta(x,y) \\ &= \bigvee_{y\in Y} \zeta(x,y) \mathbin{\&} \zeta^*(y,x) \leqslant \bigvee_{y\in Y} (\zeta \circ \zeta^*)(y,y) \leqslant \bigvee_{y\in Y} \mathrm{id}_Y(y,y) = k, \end{aligned}$$

and the second equality holds because

$$\zeta(x,z) \mathbin{\&} \zeta^*(y,x) \leqslant (\zeta \circ \zeta^*)(y,z) \leqslant \mathrm{id}_Y(y,z) = \bot. \qquad \square$$

Given a Q-map $\zeta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, it is obvious that the restriction of $\zeta$ on any (crisp) subset $W \subseteq X$ induces a Q-map $\zeta|W\colon W \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. Moreover, $\zeta$ can also be extended to any (crisp) superset $Z \supseteq Y$:

**Lemma 3.5.** *Let $\zeta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ be a* Q*-map, and let $Z \supseteq Y$. Then*

$$\tilde{\zeta}\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z, \quad \tilde{\zeta}(x,z) = \begin{cases} \zeta(x,z) & \textit{if } z \in Y, \\ \bot & \textit{if } z \notin Y \end{cases}$$

*is the unique* Q*-map from $X$ to $Z$ satisfying*

$$\tilde{\zeta}(x,y) = \zeta(x,y) \quad \textit{and} \quad (\tilde{\zeta})^*(y,x) = \zeta^*(y,x) \tag{3.iv}$$

*for all $x \in X$, $y \in Y$.*

*Proof.* It is easy to see that $\tilde{\zeta}$ and

$$\eta\colon Z \relbar\joinrel\mapstochar\joinrel\rightarrow X, \quad \eta(z,x) = \begin{cases} \zeta^*(z,x) & \text{if } z \in Y, \\ \bot & \text{if } z \notin Y \end{cases}$$

satisfy (3.ii) and (3.iii), and thus $\tilde{\zeta} \dashv \eta$, making $\tilde{\zeta}$ a Q-map. For the uniqueness of $\tilde{\zeta}$, let $\zeta'\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ be another Q-map satisfying (3.iv). Then for any $x \in X$ and $z \in Z \setminus Y$,

$$\begin{aligned} \zeta'(x,z) &\leqslant \zeta'(x,z) \mathbin{\&} \Big(\bigvee_{y\in Y} \zeta^*(y,x) \mathbin{\&} \zeta(x,y)\Big) && \text{(Inequality (3.ii))} \\ &= \bigvee_{y\in Y} \zeta'(x,z) \mathbin{\&} \zeta^*(y,x) \mathbin{\&} \zeta(x,y) \\ &= \bigvee_{y\in Y} \zeta'(x,z) \mathbin{\&} \zeta'^*(y,x) \mathbin{\&} \zeta(x,y) && \text{(Equations (3.iv))} \\ &\leqslant \bigvee_{y\in Y} \mathrm{id}_Z(y,z) \mathbin{\&} \zeta(x,y) && \text{(Inequality (3.iii))} \\ &= \bigvee_{y\in Y} \bot \mathbin{\&} \zeta(x,y) \\ &= \bot, \end{aligned}$$

which shows that $\zeta' = \tilde{\zeta}$. $\square$

Given (crisp) sets $X$, $Y$, we say that a $\mathsf{Q}$-map $\zeta\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ is *symmetric* [10] if $\zeta^* = \zeta^{\mathrm{op}}$, i.e., if

$$\zeta^*(y,x) = \zeta(x,y)$$

for all $x \in X$, $y \in Y$. In particular, every map $f\colon X \longrightarrow Y$ between (crisp) sets induces a symmetric $\mathsf{Q}$-map

$$f_\circ\colon X \to\!\!\!\!\!\!\mapsto Y,\quad f_\circ(x,y) = \begin{cases} k & \text{if } y = f(x),\\ \bot & \text{else,}\end{cases}$$

called the *graph* of $f$. Thus we have a faithful functor

$$(-)_\circ\colon \mathbf{Set} \longrightarrow \mathsf{Q}\text{-}\mathbf{Map}.$$

**Lemma 3.6.** *Let $\zeta\colon X \to\!\!\!\!\!\!\mapsto Y$ be a $\mathsf{Q}$-map. Then $\zeta$ is symmetric if, and only if, the $\mathsf{Q}$-relation*

$$\zeta_{\mathsf{s}}\colon X \to\!\!\!\!\!\!\mapsto Y,\quad \zeta_{\mathsf{s}}(x,y) = \zeta(x,y) \wedge \zeta^*(y,x)$$

*is a symmetric $\mathsf{Q}$-map.*

*Proof.* If $\zeta$ is symmetric, then $\zeta^*(y,x) = \zeta^{\mathrm{op}}(y,x) = \zeta(x,y)$, and consequently $\zeta_{\mathsf{s}}(x,y) = \zeta(x,y)$ for all $x \in X$, $y \in Y$. Thus $\zeta_{\mathsf{s}} = \zeta$ is a symmetric $\mathsf{Q}$-map. Conversely, if $\zeta_{\mathsf{s}}$ is a symmetric $\mathsf{Q}$-map, then

$$\begin{aligned}
\zeta(x,y) &\leqslant (\zeta \circ \zeta_{\mathsf{s}}^* \circ \zeta_{\mathsf{s}})(x,y) && (\zeta_{\mathsf{s}} \dashv \zeta_{\mathsf{s}}^*)\\
&= \bigvee_{w\in X}\bigvee_{z\in Y} \zeta(w,y) \,\&\, \zeta_{\mathsf{s}}(w,z) \,\&\, \zeta_{\mathsf{s}}(x,z) && (\zeta_{\mathsf{s}} \text{ is symmetric})\\
&\leqslant \bigvee_{w\in X}\bigvee_{z\in Y} \zeta(w,y) \,\&\, \zeta^*(z,w) \,\&\, \zeta_{\mathsf{s}}(x,z)\\
&\leqslant \bigvee_{z\in Y} \mathrm{id}_Y(z,y) \,\&\, \zeta_{\mathsf{s}}(x,z) && (\zeta \dashv \zeta^*)\\
&= \zeta_{\mathsf{s}}(x,y)\\
&\leqslant \zeta(x,y)
\end{aligned}$$

for all $x \in X$, $y \in Y$. Thus $\zeta = \zeta_{\mathsf{s}}$ is a symmetric $\mathsf{Q}$-map. □

The aim of this section is to answer the following questions:

- Is every $\mathsf{Q}$-map symmetric?
- Is every $\mathsf{Q}$-map the graph of a map in **Set**?

In fact, both the answers are negative, and we will provide necessary and sufficient conditions on $\mathsf{Q}$ for them to be true.

**Definition 3.7.** Let $\mathsf{Q}$ be a non-trivial, commutative and unital quantale. We say that:

(1) $\mathsf{Q}$ is *lean*, if

$$(p \vee q = k \text{ and } p \,\&\, q = \bot) \implies (p = k \text{ or } q = k) \tag{3.v}$$

and

$$p \,\&\, q = k \iff p = q = k \tag{3.vi}$$

for all $p, q \in \mathsf{Q}$;

(2) $\mathsf{Q}$ is *weakly lean*, if

$$\Big(\bigvee_{i\in I} p_i \,\&\, q_i = k \text{ and } p_i \,\&\, q_j = \bot\ (i \neq j)\Big) \implies \Big(k \leqslant \bigvee_{i\in I} (p_i \wedge q_i) \,\&\, (p_i \wedge q_i)\Big) \tag{3.vii}$$

for all $p_i, q_i \in \mathsf{Q}$ ($i \in I$).

**Remark 3.8.** (1) Note that (3.vi) is always true when Q is integral. Indeed, $p \mathbin{\&} q = k$ implies that

$$k = p \mathbin{\&} q \leqslant p \mathbin{\&} k = p \leqslant k,$$

which forces $p = k$, and $q = k$ follows from similar calculations. Hence, Definition 3.7(1) generalizes the notion of *lean* in [12, Section III.1.2], where the quantale Q is required to be integral, and consequently, only (3.v) is presented there.

(2) The condition (3.vi) is a modification of the condition given in [10, Proposition 3.5.3(3)]. The difference between our following Theorem 3.12 and [10, Proposition 3.5.3] is that we deal with the discrete case, i.e., Q-maps between (crisp) sets (instead of Q-*categories* as in [10]).

**Lemma 3.9.** *If* Q *is lean, then* Q *is weakly lean.*

*Proof.* Suppose that $p_i, q_i \in \mathsf{Q}$ $(i \in I)$, $\bigvee\limits_{i\in I} p_i \mathbin{\&} q_i = k$ and $p_i \mathbin{\&} q_j = \bot$ $(i \neq j)$. Since $\bot < k$, there exists $j \in I$ such that $p_j \mathbin{\&} q_j > \bot$. Let

$$u := p_j \mathbin{\&} q_j \quad \text{and} \quad v := \bigvee_{\substack{i\in I\\ i\neq j}} p_j \mathbin{\&} q_j.$$

Then

$$u \vee v = k \quad \text{and} \quad u \mathbin{\&} v = p_j \mathbin{\&} q_j \mathbin{\&} \Big(\bigvee_{\substack{i\in I\\ i\neq j}} p_i \mathbin{\&} q_i\Big) = \bigvee_{\substack{i\in I\\ i\neq j}} p_j \mathbin{\&} q_j \mathbin{\&} p_i \mathbin{\&} q_i = \bot. \tag{3.viii}$$

Note that $v \neq k$. Indeed, if $v = k$, then $u \mathbin{\&} v = u \mathbin{\&} k = u > \bot$, contradicting to (3.viii). Thus, since Q is lean, we deduce from (3.viii) that $u = p_j \mathbin{\&} q_j = k$, and consequently $p_j = q_j = k$. It follows that

$$\bigvee_{i\in I} (p_i \wedge q_i) \mathbin{\&} (p_i \wedge q_i) \geqslant (p_j \wedge q_j) \mathbin{\&} (p_j \wedge q_j) = (k \wedge k) \mathbin{\&} (k \wedge k) = k,$$

showing that Q is weakly lean. □

**Lemma 3.10.** *If* Q *is integral, then* Q *is weakly lean.*

*Proof.* Suppose that $p_i, q_i \in \mathsf{Q}$ $(i \in I)$, $\bigvee\limits_{i\in I} p_i \mathbin{\&} q_i = k$ and $p_i \mathbin{\&} q_j = \bot$ $(i \neq j)$. Note that for each $i \in I$,

$$\begin{aligned} p_i &= p_i \mathbin{\&} k \\ &= p_i \mathbin{\&} \Big(\bigvee_{j\in I} p_j \mathbin{\&} q_j\Big) \\ &= \bigvee_{j\in I} p_i \mathbin{\&} p_j \mathbin{\&} q_j \\ &= (p_i \mathbin{\&} p_i \mathbin{\&} q_i) \vee \Big(\bigvee_{\substack{j\in I\\ j\neq i}} p_i \mathbin{\&} p_j \mathbin{\&} q_j\Big) \\ &= (p_i \mathbin{\&} p_i \mathbin{\&} q_i) \vee \bot \\ &= p_i \mathbin{\&} p_i \mathbin{\&} q_i. \end{aligned}$$

Since Q is integral, we have $p \mathbin{\&} q \leqslant (p \mathbin{\&} k) \wedge (k \mathbin{\&} q) = p \wedge q$ for all $p, q \in \mathsf{Q}$, and consequently

$$\bigvee_{i\in I} (p_i \wedge q_i) \mathbin{\&} (p_i \wedge q_i) \geqslant \bigvee_{i\in I} p_i \mathbin{\&} q_i \mathbin{\&} p_i \mathbin{\&} q_i = \bigvee_{i\in I} p_i \mathbin{\&} q_i = k,$$

as desired. □

**Example 3.11.** For the examples listed in 2.1:

(1) $(C_3, \&, k)$ is a non-integral lean quantale.

(2) Every frame is divisible, and thus integral. Then it follows from Lemma 3.10 that every frame is weakly lean. Moreover, $F_1 = \{\bot, p, q, \top\}$ is not lean while $F_2 = \{\bot, p, q, r, \top\}$ is lean. Therefore, a weakly lean quantale need not be lean.

(3) The quantale $(M_3, \&, k)$ is not lean.

(4) For each continuous t-norm $*$ on $[0, 1]$, the quantale $([0, 1], *, 1)$ is lean.

(5) The quantale $([-\infty, \infty], +, 0)$ is not weakly lean, while the Lawvere quantale $([0, \infty], +, 0)$ is lean.

(6) Consider the free quantale $(\mathsf{P}M, \&, \{k\})$ induced by a commutative monoid $(M, \&, k)$:

- $(\mathsf{P}M, \&, \{k\})$ is lean if, and only if, $k$ is the only element of $M$ with an inverse; that is, if $m \in M$ and $m \neq k$, then there exists no $m' \in M$ with $m \,\&\, m' = k$.
- $(\mathsf{P}M, \&, \{k\})$ is weakly lean if, and only if, there exist no $m, m' \in M$ such that $m \neq m'$ and $m \,\&\, m' = k$.

In particular, the free quantale induced by the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ (under the usual addition modulo 2) is non-integral, weakly lean, but not lean.

**Theorem 3.12.** *Let $\mathsf{Q}$ be a non-trivial, commutative and unital quantale. Then every $\mathsf{Q}$-map is symmetric if, and only if, $\mathsf{Q}$ is weakly lean.*

*Proof.* For the necessity, suppose that every $\mathsf{Q}$-map is symmetric. Assume that $p_i, q_i \in \mathsf{Q}$ $(i \in I)$, $\bigvee\limits_{i \in I} p_i \,\&\, q_i = k$ and $p_i \,\&\, q_j = \bot$ $(i \neq j)$. Define a $\mathsf{Q}$-relation

$$\zeta \colon \{\star\} \to\!\!\!\!\!\!\mapsto I, \quad \zeta(\star, i) = p_i,$$

where $\{\star\}$ refers to the singleton set. Note that for each $i \in I$, from $p_i \,\&\, q_i \leqslant k$ and $q_i \,\&\, p_j = \bot$ $(j \neq i)$ we have

$$\begin{aligned}
q_i &= q_i \wedge q_i \\
&\leqslant \Big(\bigwedge_{\substack{j \in I \\ j \neq i}} p_j \to \bot\Big) \wedge (p_i \to k) \\
&= \Big(\bigwedge_{\substack{j \in I \\ j \neq i}} \zeta(\star, j) \to \bot\Big) \wedge (\zeta(\star, i) \to k) \\
&= \bigwedge_{j \in I} \zeta(\star, j) \to \mathrm{id}_I(i, j) \\
&= (\zeta \searrow \mathrm{id}_I)(i, \star) \\
&= \Big(\bigvee_{j \in I} p_j \,\&\, q_j\Big) \,\&\, \Big(\bigwedge_{l \in I} \zeta(\star, l) \to \mathrm{id}_I(i, l)\Big) \\
&= \Big[\bigvee_{\substack{j \in I \\ j \neq i}} p_j \,\&\, q_j \,\&\, \Big(\bigwedge_{l \in I} p_l \to \mathrm{id}_I(i, l)\Big)\Big] \vee \Big[p_i \,\&\, q_i \,\&\, \Big(\bigwedge_{l \in I} p_l \to \mathrm{id}_I(i, l)\Big)\Big] \\
&\leqslant \Big[\bigvee_{\substack{j \in I \\ j \neq i}} p_j \,\&\, q_j \,\&\, (p_j \to \bot)\Big] \vee \Big[p_i \,\&\, q_i \,\&\, (p_i \to k)\Big] \\
&\leqslant \bot \vee (q_i \,\&\, k) \\
&= q_i.
\end{aligned}$$

Thus $q_i = (\zeta \searrow \mathrm{id}_I)(i,\star) = \zeta^*(i,\star)$. It follows that

$$(\zeta^* \circ \zeta)(\star,\star) = \bigvee_{i\in I} \zeta^*(i,\star) \mathbin{\&} \zeta(\star,i) = \bigvee_{i\in I} q_i \mathbin{\&} p_i = k = \mathrm{id}_{\{\star\}}(\star,\star),$$

which in conjunction with Lemma 2.3 shows that $\zeta$ is a $\mathsf{Q}$-map. Since every $\mathsf{Q}$-map is symmetric, in particular we have

$$q_i = \zeta^*(i,\star) = \zeta^{\mathrm{op}}(i,\star) = \zeta(\star,i) = p_i$$

for all $i \in I$, and consequently

$$\bigvee_{i\in I}(p_i \wedge q_i) \mathbin{\&} (p_i \wedge q_i) = \bigvee_{i\in I}(p_i \wedge p_i) \mathbin{\&} (q_i \wedge q_i) = \bigvee_{i\in I} p_i \mathbin{\&} q_i = k.$$

Hence $\mathsf{Q}$ is weakly lean.

Conversely, for the sufficiency, suppose that $\mathsf{Q}$ is weakly lean. In order to show that every $\mathsf{Q}$-map $\zeta\colon X \to\!\!\!\!\!\!\mapsto Y$ is symmetric, by Lemma 3.6 it suffices to show that $\zeta_{\mathsf{s}}$ is always a symmetric $\mathsf{Q}$-map. To this end, let $x \in X$. First,

$$k = \mathrm{id}_X(x,x) \leqslant (\zeta^* \circ \zeta)(x,x) = \bigvee_{y\in Y} \zeta^*(y,x) \mathbin{\&} \zeta(x,y) \leqslant \bigvee_{y\in Y} \mathrm{id}_Y(y,y) = k,$$

Second, for any $y,z \in Y$ with $y \neq z$,

$$\zeta(x,z) \mathbin{\&} \zeta^*(y,x) \leqslant (\zeta \circ \zeta^*)(y,z) \leqslant \mathrm{id}_Y(y,z) = \bot.$$

Thus

$$\bigvee_{y\in Y} \zeta^*(y,x) \mathbin{\&} \zeta(x,y) = k \quad \text{and} \quad \zeta(x,z) \mathbin{\&} \zeta^*(y,x) = \bot \ (y \neq z) \tag{3.ix}$$

Since $\mathsf{Q}$ is weakly lean, (3.ix) forces

$$\zeta_{\mathsf{s}}^{\mathrm{op}} \circ \zeta_{\mathsf{s}}(x,x) = \bigvee_{y\in Y} \zeta_{\mathsf{s}}^{\mathrm{op}}(y,x) \mathbin{\&} \zeta_{\mathsf{s}}(x,y) = \bigvee_{y\in Y}(\zeta(x,y) \wedge \zeta^*(y,x)) \mathbin{\&} (\zeta(x,y) \wedge \zeta^*(y,x)) \geqslant k = \mathrm{id}_X(x,x).$$

Moreover,

$$\zeta_{\mathsf{s}} \circ \zeta_{\mathsf{s}}^{\mathrm{op}}(y,z) = \bigvee_{x\in X} \zeta_{\mathsf{s}}(x,z) \mathbin{\&} \zeta_{\mathsf{s}}^{\mathrm{op}}(y,x) \leqslant \bigvee_{x\in X} \zeta(x,z) \mathbin{\&} \zeta^*(y,x) \leqslant \mathrm{id}_Y(y,z)$$

for all $y,z \in Y$. Thus $\zeta_{\mathsf{s}} \dashv \zeta_{\mathsf{s}}^{\mathrm{op}}$; that is, $\zeta_{\mathsf{s}}$ is a symmetric $\mathsf{Q}$-map. □

**Corollary 3.13.** *If* $\mathsf{Q}$ *is weakly lean, then for any* $\mathsf{Q}$*-maps* $\zeta,\eta\colon X \to\!\!\!\!\!\!\mapsto Y$*,* $\zeta = \eta$ *whenever* $\zeta \leqslant \eta$*.*

*Proof.* First, $\zeta \leqslant \eta$ implies that $\eta^* \leqslant \zeta^*$. Second, since $\mathsf{Q}$ is weakly lean, Theorem 3.12 indicates that $\eta^{\mathrm{op}} = \eta^* \leqslant \zeta^* = \zeta^{\mathrm{op}}$. Thus $\eta \leqslant \zeta$, as desired. □

**Remark 3.14.** Following the terminology of [11, Definition 3.1(2)], Corollary 3.13 actually says that $\mathsf{Q}$-**Map** is *map-discrete* when $\mathsf{Q}$ is weakly lean.

We would also like to point out that every $\mathsf{Q}$-map is symmetric if $\mathsf{Q}$ is *modular* (cf. [10, Definition 2.5.1]) and *localic* (cf. [10, Definition 2.1.1]), which is an immediate consequence of [10, Propositions 2.5.3(4) and 2.5.9]. However, this is only a sufficient condition, since a weakly lean quantale need not be localic; for example, the quantale $M_3$ given in Example 2.1(3) is lean but not localic.

**Theorem 3.15.** *Let* $\mathsf{Q}$ *be a non-trivial, commutative and unital quantale. Then every* $\mathsf{Q}$*-map is the graph of a map in* **Set** *if, and only if,* $\mathsf{Q}$ *is lean. In this case,* **Set** *and* $\mathsf{Q}$-**Map** *are isomorphic categories.*

*Proof.* For the necessity, suppose that every Q-map is the graph of a map in **Set**.

First, let $p, q \in Q$ with $p \vee q = k$ and $p \,\&\, q = \bot$. Then both $p$ and $q$ are idempotent, since

$$p = p \,\&\, k = p \,\&\, (p \vee q) = (p \,\&\, p) \vee (p \,\&\, q) = (p \,\&\, p) \vee \bot = p \,\&\, p \quad \text{and}$$
$$q = k \,\&\, q = (p \vee q) \,\&\, q = (p \,\&\, q) \vee (q \,\&\, q) = \bot \vee (q \,\&\, q) = q \,\&\, q.$$

It follows that

$$p \leqslant p \to k, \quad q \leqslant q \to k, \quad p \leqslant q \to \bot \quad \text{and} \quad q \leqslant p \to \bot. \tag{3.x}$$

Define a Q-relation

$$\zeta \colon \{\star\} \longrightarrow\!\!\!\!\!\!\mapsto \{p, q\}, \quad \zeta(\star, p) = p, \quad \zeta(\star, q) = q.$$

Then, from (3.x) we deduce that

$$\zeta^*(p, \star) = (\zeta \searrow \mathrm{id}_Y)(p, \star) = (\zeta(\star, p) \to \mathrm{id}_Y(p, p)) \wedge (\zeta(\star, q) \to \mathrm{id}_Y(p, q)) = (p \to k) \wedge (q \to \bot) \geqslant p \wedge p = p,$$

and similarly $\zeta^*(q, \star) \geqslant q$. Thus

$$(\zeta^* \circ \zeta)(\star, \star) = (\zeta^*(p, \star) \,\&\, \zeta(\star, p)) \vee (\zeta^*(q, \star) \,\&\, \zeta(\star, q)) \geqslant (p \,\&\, p) \vee (q \,\&\, q) = p \vee q = k.$$

By Lemma 2.3, $\zeta$ is a Q-map, and hence the graph of a map $f \colon \{\star\} \longrightarrow \{p, q\}$ in **Set**, which forces $p = k$ or $q = k$.

Second, let $p, q \in Q$ with $p \,\&\, q = k$. Define a Q-relation

$$\eta \colon \{\star\} \longrightarrow\!\!\!\!\!\!\mapsto \{\star\}, \quad \eta(\star, \star) = p.$$

Since

$$(\eta^* \circ \eta)(\star, \star) = (\eta(\star, \star) \to \mathrm{id}_{\{\star\}}(\star, \star)) \,\&\, \eta(\star, \star) = (p \to k) \,\&\, p \geqslant q \,\&\, p = k,$$

$\eta$ is a Q-map by Lemma 2.3. Thus $\eta$ is a graph of the unique (identity) map $g \colon \{\star\} \longrightarrow \{\star\}$, which forces $p = k$, and consequently $q = q \,\&\, k = q \,\&\, p = k$.

Conversely, for the sufficiency, suppose that Q is lean. Let $\zeta \colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ be a Q-map. For each $x \in X$, since

$$k = \mathrm{id}_X(x, x) \leqslant (\zeta^* \circ \zeta)(x, x) = \bigvee_{y \in Y} \zeta^*(y, x) \,\&\, \zeta(x, y) \leqslant \bigvee_{y \in Y} (\zeta \circ \zeta^*)(y, y) \leqslant \bigvee_{y \in Y} \mathrm{id}_Y(y, y) = k, \tag{3.xi}$$

we have

$$\bigvee_{y \in Y} \zeta^*(y, x) \,\&\, \zeta(x, y) = k.$$

Since $k > \bot$, there exists $f(x) \in Y$ with $p := \zeta^*(f(x), x) \,\&\, \zeta(x, f(x)) > \bot$. Let

$$q := \bigvee_{\substack{y \in Y \\ y \neq f(x)}} \zeta^*(y, x) \,\&\, \zeta(x, y).$$

Then $p \vee q = k$, and

$$\begin{aligned}
p \,\&\, q &= \zeta^*(f(x), x) \,\&\, \zeta(x, f(x)) \,\&\, \Big( \bigvee_{\substack{y \in Y \\ y \neq f(x)}} \zeta^*(y, x) \,\&\, \zeta(x, y) \Big) \\
&= \bigvee_{\substack{y \in Y \\ y \neq f(x)}} \zeta^*(f(x), x) \,\&\, \zeta(x, f(x)) \,\&\, \zeta^*(y, x) \,\&\, \zeta(x, y) \\
&= \bigvee_{\substack{y \in Y \\ y \neq f(x)}} \zeta(x, f(x)) \,\&\, \bot \,\&\, \zeta^*(f(x), x)) \\
&= \bot,
\end{aligned}$$

where the third equality follows from

$$\zeta(x,z) \,\&\, \zeta^*(y,x) \leqslant (\zeta \circ \zeta^*)(y,z) \leqslant \mathrm{id}_Y(y,z) = \bot$$

for all $y, z \in Y$ with $y \neq z$.

Note that $q \neq k$, because $q = k$ would lead to $p \,\&\, q = p \,\&\, k = p > \bot$, contradicting to $p \,\&\, q = \bot$. Therefore, given that $\mathsf{Q}$ is lean, we conclude that $p = \zeta^*(f(x), x) \,\&\, \zeta(x, f(x)) = k$, and consequently

$$\zeta^*(f(x), x) = \zeta(x, f(x)) = k.$$

We claim that $\zeta$ is the graph of the map $f\colon X \longrightarrow Y$. Indeed, it is obvious that $f_\circ \leqslant \zeta$, and hence the conclusion follows immediately from Lemma 3.9 and Corollary 3.13. □

As mentioned in Remark 3.8(1), Theorem 3.15 reduces to [12, Proposition III.1.2.1] when $\mathsf{Q}$ is integral:

**Corollary 3.16.** *(See [12].) Let $\mathsf{Q}$ be a non-trivial, commutative and integral quantale. Then every $\mathsf{Q}$-map is the graph of a map in* **Set** *if, and only if,*

$$(p \vee q = \top \text{ and } p \,\&\, q = \bot) \implies (p = \top \text{ or } q = \top)$$

*for $p, q \in \mathsf{Q}$.*

## 4. Quantale-valued partial maps

Note that coproducts in the category $\mathsf{Q}$-**Map** are given by disjoint unions of sets as in **Set**:

**Lemma 4.1.** $\mathsf{Q}$-**Map** *has all coproducts.*

*Proof.* Let $X = \coprod\limits_{i \in I} X_i$ be the disjoint union of a family of sets $\{X_i\}_{i \in I}$, and let $\iota_i\colon X_i \relbar\joinrel\mapstochar\joinrel\rightarrow X$ $(i \in I)$ be the graphs of the inclusion maps. For any family of $\mathsf{Q}$-maps $\{\zeta_i\colon X_i \relbar\joinrel\mapstochar\joinrel\rightarrow Y\}_{i \in I}$, define a $\mathsf{Q}$-relation

$$\zeta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y, \quad \zeta(x,y) = \zeta_i(x,y) \quad \text{if } x \in X_i,\ y \in Y.$$

Then $\zeta$ is clearly a $\mathsf{Q}$-map, and $\zeta \circ \iota_i = \zeta_i$ $(i \in I)$ since

$$\begin{aligned}
(\zeta \circ \iota_i)(x,y) &= \bigvee_{w \in X} \zeta(w,y) \,\&\, \iota_i(x,w) \\
&= \Big( \bigvee_{w \in X \setminus X_i} \zeta(w,y) \,\&\, \bot \Big) \vee \Big( \bigvee_{w \in X_i} \zeta(w,y) \,\&\, \mathrm{id}_{X_i}(x,w) \Big) \\
&= \bigvee_{w \in X_i} \zeta_i(w,y) \,\&\, \mathrm{id}_{X_i}(x,w) \\
&= \zeta_i(x,y)
\end{aligned}$$

for all $x \in X_i$, $y \in Y$. Finally, for the uniqueness of $\zeta$, let $\eta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ be a $\mathsf{Q}$-map satisfying $\eta \circ \iota_i = \zeta_i$ for all $i \in I$. Then

$$\zeta_i(x,y) = \eta \circ \iota_i(x,y) = \bigvee_{w \in X} \eta(w,y) \,\&\, \mathrm{id}_X(x,w) = \eta(x,y)$$

for all $x \in X_i$, $y \in Y$, which completes the proof. □

**Lemma 4.2.** *If $\zeta\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $\eta\colon X' \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$ are $\mathsf{Q}$-maps, then*

$$\zeta \amalg \eta\colon X \amalg X' \relbar\joinrel\mapstochar\joinrel\rightarrow Y \amalg Y'$$

*is a $\mathsf{Q}$-map.*

*Proof.* It is straightforward to check that $\zeta^* \amalg \eta^* \colon Y \amalg Y' \longrightarrow\!\!\!\!\!\!\!+\ X \amalg X'$ is the right adjoint of $\zeta \amalg \eta$ in Q-**Rel**. □

Let

$$X_+ := X \amalg \{\star\}$$

denote the disjoint union of every (crisp) set $X$ and the singleton set $\{\star\}$.

**Definition 4.3.** Let $X, Y$ be (crisp) sets. A *partial* Q*-map* $\zeta$ from $X$ to $Y$ is a Q-map $\zeta \colon X \looparrowright Y_+$.

**Remark 4.4.** The singleton set $\{\star\}$ may not be a terminal object of Q-**Map**. For example, if $\mathsf{Q} = M_3$ (see Example 2.1(3)), then there are two different Q-maps from $\{\star\}$ to $\{\star\}$:

- the graph of the identity map on $\{\star\}$;
- the Q-map

  $$\zeta \colon \{\star\} \looparrowright \{\star\}, \quad \zeta(\star,\star) = a,$$

  whose right adjoint is given by

  $$\eta \colon \{\star\} \looparrowright \{\star\}, \quad \eta(\star,\star) = b.$$

**Remark 4.5.** When Q is an integral quantale, every partial Q-map is a *fuzzy partial function* defined in [4, 5], but not vice versa.

In order to construct the category of sets and partial Q-maps, note that by setting

$$\mathcal{C} = \mathsf{Q}\text{-}\mathbf{Map} \quad \text{and} \quad A = \{\star\}$$

in (1.i) and (1.ii) we obtain a pair of adjoint functors

$$\mathsf{Q}\text{-}\mathbf{Map} \underset{\mathsf{U}}{\overset{\mathsf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \{\star\}/\mathsf{Q}\text{-}\mathbf{Map} \tag{4.i}$$

given by the following data:

- U is the forgetful functor;
- the functor F sends each set $X$ to the graph of the inclusion map $\{\star\} \hookrightarrow X_+$, and sends each Q-map $\zeta \colon X \looparrowright Y$ to the Q-map

  $$\zeta_+ := \zeta \amalg \mathrm{id}_{\{\star\}} \colon X_+ \looparrowright Y_+;$$

- for each set $X$, the component of the unit $\iota_X \colon X \looparrowright X_+$ is the graph of the inclusion map $X \hookrightarrow X_+$;
- for each object $\mu \colon \{\star\} \looparrowright X$ in $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$, the component of the counit $\varepsilon_\mu \colon \mathsf{F}X \looparrowright \mu$ is given by

  $$\varepsilon_\mu(x,y) = \begin{cases} \mathrm{id}_X(x,y) & \text{if } x,y \in X, \\ \mu(\star,y) & \text{if } x = \star \text{ and } y \in X. \end{cases}$$

The adjunction $\mathsf{F} \dashv \mathsf{U}$ induces the *maybe monad* $(\mathsf{T},\mathsf{m},\iota)$ on Q-**Map**, whose endofunctor $\mathsf{T} = \mathsf{UF}$ carries each set $X$ to $X_+$, and whose multiplication is given by

$$\mathsf{m}_X \colon (X_+)_+ \looparrowright X_+, \quad \mathsf{m}_X(x,y) = \begin{cases} \mathrm{id}_X(x,y) & \text{if } x,y \in X, \\ k & \text{if } x = y = \star, \\ \bot & \text{if } x = \star,\ y \in X \text{ or } x \in X,\ y = \star. \end{cases} \tag{4.ii}$$

The isomorphism between $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$ and the Eilenberg-Moore category $\mathsf{Q}\text{-}\mathbf{Map}^{\mathsf{T}}$ is not difficult to be observed. Indeed, a T-algebra $(X,\mu)$ consists of a set $X$ and a Q-map $\mu \colon X_+ \looparrowright X$ such that

$$\mu \circ \iota_X = \mathrm{id}_X,$$

which is clearly uniquely determined by an object in $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$. Moreover, it is easy to see that a morphism $\zeta\colon (X,\mu) \longrightarrow\!\!\!\!\!+ (Y,\lambda)$ of $\mathsf{T}$-algebras is a $\mathsf{Q}$-map $\zeta\colon X \longrightarrow\!\!\!\!\!+ Y$ such that

$$(\zeta\circ\mu)(\star,y) = \lambda(\star,y)$$

for all $y\in Y$, which is simply a morphism in $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$. Therefore, in what follows we identify $\mathsf{Q}\text{-}\mathbf{Map}^{\mathsf{T}}$ with $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$.

Furthermore, objects in the Kleisli category $\mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$ are sets, and a morphism from $X$ to $Y$ in $\mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$ is exactly a $\mathsf{Q}$-map $X \longrightarrow\!\!\!\!\!+ Y_+$; that is, a partial $\mathsf{Q}$-map from $X$ to $Y$. Thus, we denote by

$$\mathsf{Q}\text{-}\mathbf{Map}^{\partial} := \mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$$

the category of sets and partial $\mathsf{Q}$-maps. By Lemma 3.5, $\mathsf{Q}\text{-}\mathbf{Map}$ is embedded in $\mathsf{Q}\text{-}\mathbf{Map}^{\partial}$ as a full subcategory.

**Lemma 4.6.** *The composite of partial* $\mathsf{Q}$*-maps* $\zeta\colon X \longrightarrow\!\!\!\!\!+ Y_+$ *and* $\eta\colon Y \longrightarrow\!\!\!\!\!+ Z_+$ *is given by*

$$(\eta\diamond\zeta)(x,z) = \begin{cases} \bigvee\limits_{y\in Y} \eta(y,z) \;\&\; \zeta(x,y) & \text{if } x\in X \text{ and } z\in Z,\\ \Big(\bigvee\limits_{y\in Y} \eta(y,\star) \;\&\; \zeta(x,y)\Big) \vee \zeta(x,\star) & \text{if } x\in X \text{ and } z=\star. \end{cases} \tag{4.iii}$$

*Proof.* (4.iii) follows from $\eta\diamond\zeta = \mathsf{m}_Z\circ(\mathsf{T}\eta)\circ\zeta$ and

$$(\mathsf{m}_Z\circ(\mathsf{T}\eta)\circ\zeta)(x,z) = \bigvee_{w\in(Z_+)_+}\bigvee_{y\in Y_+} \mathsf{m}_Z(w,z) \;\&\; (\mathsf{T}\eta)(y,w) \;\&\; \zeta(x,y) = \bigvee_{y\in Y} \eta(y,z) \;\&\; \zeta(x,y),$$

$$(\mathsf{m}_Z\circ(\mathsf{T}\eta)\circ\zeta)(x,\star) = \bigvee_{z'\in(Z_+)_+}\bigvee_{y\in Y_+} \mathsf{m}_Z(z',\star) \;\&\; (\mathsf{T}\eta)(y,z') \;\&\; \zeta(x,y) = \Big(\bigvee_{y\in Y} \eta(y,\star) \;\&\; \zeta(x,y) \;\&\; k\Big) \vee (\zeta(x,\star) \;\&\; k)$$

for all $x\in X$, $z\in Z$. □

**Remark 4.7.** Let $\zeta\colon X \longrightarrow Y_+$ be a partial $\mathsf{Q}$-map from $X$ to $Y$. The value

$$\zeta(x,\star)$$

is interpreted as the degree that $x$ has no image under the map $\zeta$, and so is the value

$$\zeta^*(\star,x) = \bigwedge_{z\in Y_+} \zeta(x,z) \to \mathrm{id}_{Y_+}(\star,z),$$

because the above expression means that

- For each $z\in Y_+$, if $z$ is the image of $x$ under $\zeta$, then $z$ is equal to $\star$; in other words, $x$ has no image under $\zeta$ within $Y$.

The adjunction $\zeta\dashv\zeta^*$ can be translated as follows:

- For every $x\in X$, there may not be $y\in Y$ such that $y$ is the image of $x$ under $\zeta$; because $\mathrm{id}_X\leqslant\zeta^*\circ\zeta$ means that

  $$k\leqslant\bigvee_{y\in Y_+}\zeta^*(y,x) \;\&\; \zeta(x,y) = (\zeta^*(\star,x) \;\&\; \zeta(x,\star)) \vee \Big(\bigvee_{y\in Y}\zeta^*(y,x) \;\&\; \zeta(x,y)\Big)$$

  for all $x\in X$.

- If $y,z\in Y$ are both the images of $x$ under $\zeta$, then $y$ is equal to $z$; because $\zeta\circ\zeta^*\leqslant\mathrm{id}_{Y_+}$ means that

  $$\bigvee_{x\in X}\zeta(x,z) \;\&\; \zeta^*(y,x) \leqslant \mathrm{id}_{Y_+}(y,z) = \mathrm{id}_Y(y,z)$$

  for all $y,z\in Y$. Moreover, setting $y\in Y$ and $z=\star$ in the above inequality we obtain that

  $$\zeta(x,\star) \;\&\; \zeta^*(y,x) = \mathrm{id}_{Y_+}(y,\star) = \bot$$

  for all $x\in X$; that is, "$x$ has no image under $\zeta$" and "$y$ is the image of $x$ under $\zeta$" cannot happen simultaneously.

Given another partial Q-map $\eta\colon Y \looparrowright Z_+$ from $Y$ to $Z$, the formula (4.iii) given in Lemma 4.6 is understood as follows:

- $z \in Z$ is the image of $x$ under $\eta \diamond \zeta$ if, and only if, there exists $y \in Y$ such that $y$ is the image of $x$ under $\zeta$, and $z$ is the image of $y$ under $\eta$.
- $x$ has no image under $\eta \diamond \zeta$ if, and only if, either of the following cases holds:
  - $x$ has no image under $\zeta$;
  - $x$ has an image $y$ under $\zeta$, but $y$ has no image under $\eta$.

**Theorem 4.8.** *Let* Q *be a non-trivial, commutative and unital quantale. Then every partial* Q*-map is the graph of a partial map in* **Set** *if, and only if,* Q *is lean. In this case,* $\mathbf{Set}^{\partial}$ *and* Q-$\mathbf{Map}^{\partial}$ *are isomorphic categories.*

*Proof.* Since Q-**Map** is embedded in Q-$\mathbf{Map}^{\partial}$ as a full subcategory, the necessity is an immediate consequence of Theorem 3.15. For the sufficiency, suppose that Q is lean. Since every partial Q-map is a Q-map $\zeta\colon X \looparrowright Y_+$, there is a map $f\colon X \longrightarrow Y_+$ in **Set** such that $\zeta = f_\natural$, which is exactly a partial map in **Set**. □

Let $(X_+, \tau_X) \in \{\star\}/$Q-**Map** denote the *free* T-algebra on a set $X$. Then

$$\tau_X\colon \{\star\} \looparrowright X_+, \quad \tau_X(\star, x) = \begin{cases} k & \text{if } x = \star, \\ \bot & \text{if } x \in X \end{cases} \tag{4.iv}$$

is the restriction of $\mathsf{m}_X\colon (X_+)_+ \looparrowright X_+$ (see (4.ii)), which is exactly the graph of the inclusion map $\{\star\} \hookrightarrow X_+$. It is well known that the canonincal functor

$$\mathsf{K}\colon \text{Q-}\mathbf{Map}^{\partial}(= \text{Q-}\mathbf{Map}_{\mathsf{T}}) \longrightarrow \{\star\}/\text{Q-}\mathbf{Map}(\cong \text{Q-}\mathbf{Map}^{\mathsf{T}})$$

that sends each Q-map $\zeta\colon X \looparrowright Y_+$ (i.e., partial Q-map from $X$ to $Y$) to

$$\mathsf{K}\zeta\colon (X_+, \tau_X) \looparrowright (Y_+, \tau_Y), \quad (\mathsf{K}\zeta)(x, y) = \begin{cases} \zeta(x, y) & \text{if } x \in X,\ y \in Y, \\ k & \text{if } x = \star = y, \\ \bot & \text{else} \end{cases}$$

is fully faithful (see, e.g., [21, Lemma 5.2.13]).

**Lemma 4.9.** *Assuming the axiom of choice, every object of* $\{\star\}/$Q-**Map** *is isomorphic to a free* T*-algebra, hence the functor* K *is essentially surjective.*

*Proof.* Let $(X, \mu) \in \{\star\}/$Q-**Map**. Since $X$ is the codomain of the Q-map $\mu\colon \{\star\} \looparrowright X$, it is clear that $X \neq \varnothing$. By the axiom of choice and [9], there exists a binary operation $\cdot$ on $X$ such that $(X, \cdot)$ is a group, with $e \in X$ being its unit. Then

(G1) $x \cdot z \neq y \cdot z$ and $z \cdot x \neq z \cdot y$ for all $x, y, z \in X$ with $x \neq y$, and

(G2) $x \cdot y \neq x$ for all $x, y \in X$ with $y \neq e$.

Our strategy is to show that $(X, \mu)$ is isomorphic to

$$\mathsf{K}(X \setminus \{e\}) = ((X \setminus \{e\})_+, \tau_{X \setminus \{e\}}).$$

Explicitly, we define Q-relations

$$\zeta\colon X \looparrowright (X \setminus \{e\})_+, \quad \zeta(x, a) = \begin{cases} \mu^*(x, \star) & \text{if } x \in X,\ a = \star \\ \mu^*(x \cdot a, \star) & \text{if } x, a \in X \end{cases}$$

and

$$\eta\colon (X\setminus\{e\})_+ \longrightarrow\!\!\!\!\!\!\mapsto X,\quad \eta(a,x)=\begin{cases}\mu(\star,x) & \text{if } a=\star,\ x\in X\\ \mu(\star,x\cdot a) & \text{if } a,x\in X,\end{cases}$$

and we show that $\zeta$ and $\eta$ establish an isomorphism between $(X,\mu)$ and $((X\setminus\{e\})_+,\tau_{X\setminus\{e\}})$ in $\{\star\}/\mathsf{Q}$-**Map**.

First, $\eta\circ\zeta=\mathrm{id}_X$ and $\zeta\circ\eta=\mathrm{id}_{(X\setminus\{e\})_+}$, so that $\zeta$ and $\eta$ are both Q-maps, and they are inverses of each other. Indeed, it follows from (G1) and Lemma 3.4 that

$$(\eta\circ\zeta)(x,y)=\bigvee_{a\in(X\setminus\{e\})_+}\eta(a,y)\mathbin{\&}\zeta(x,a)=(\mu(\star,y)\mathbin{\&}\mu^*(x,\star))\vee\Big(\bigvee_{a\in X\setminus\{e\}}\mu(\star,y\cdot a)\mathbin{\&}\mu^*(x\cdot a,\star)\Big)=\bot$$

for all $x,y\in X$ with $x\neq y$, and

$$\begin{aligned}
(\eta\circ\zeta)(x,x)&=(\mu(\star,x)\mathbin{\&}\mu^*(x,\star))\vee\Big(\bigvee_{a\in X\setminus\{e\}}\mu(\star,x\cdot a)\mathbin{\&}\mu^*(x\cdot a,\star)\Big)\\
&\leqslant \mathrm{id}_X(x,x)\vee\Big(\bigvee_{a\in X\setminus\{e\}}\mathrm{id}_X(x\cdot a,x\cdot a)\Big) && (\mu\dashv\mu^*)\\
&=k\\
&=(\mu^*\circ\mu)(\star,\star) && (\text{Lemma 3.4})\\
&=(\mu(\star,x)\mathbin{\&}\mu^*(x,\star))\vee\Big(\bigvee_{\substack{y\in X\\ y\neq x}}\mu(\star,y)\mathbin{\&}\mu^*(y,\star)\Big)\\
&=(\mu(\star,x)\mathbin{\&}\mu^*(x,\star))\vee\Big(\bigvee_{\substack{y\in X\\ y\neq x}}\mu(\star,x\cdot x^{-1}\cdot y)\mathbin{\&}\mu^*(x\cdot x^{-1}\cdot y,\star)\Big)\\
&\leqslant(\mu(\star,x)\mathbin{\&}\mu^*(x,\star))\vee\Big(\bigvee_{a\in X\setminus\{e\}}\mu(\star,x\cdot a)\mathbin{\&}\mu^*(x\cdot a,\star)\Big) && (x\neq y\implies x^{-1}\cdot y\neq e)\\
&=(\eta\circ\zeta)(x,x)
\end{aligned}$$

for all $x\in X$. Thus $\eta\circ\zeta=\mathrm{id}_X$. In order to show that $\zeta\circ\eta=\mathrm{id}_{(X\setminus\{e\})_+}$, from (G1), (G2) and Lemma 3.4 we deduce that

$$\begin{aligned}
(\zeta\circ\eta)(a,b)&=\bigvee_{x\in X}\zeta(x,b)\mathbin{\&}\eta(a,x)=\bigvee_{x\in X}\mu^*(x\cdot b,\star)\mathbin{\&}\mu(\star,x\cdot a)=\bot,\\
(\zeta\circ\eta)(a,\star)&=\bigvee_{x\in X}\zeta(x,\star)\mathbin{\&}\eta(a,x)=\bigvee_{x\in X}\mu^*(x,\star)\mathbin{\&}\mu(\star,x\cdot a)=\bot,\\
(\zeta\circ\eta)(\star,b)&=\bigvee_{x\in X}\zeta(x,b)\mathbin{\&}\eta(\star,x)=\bigvee_{x\in X}\mu^*(x\cdot b,\star)\mathbin{\&}\mu(\star,x)=\bot
\end{aligned}$$

for all $a,b\in X\setminus\{e\}$ with $a\neq b$. Moreover, it follows from Lemma 3.4 that

$$(\zeta\circ\eta)(\star,\star)=\bigvee_{x\in X}\zeta(x,\star)\mathbin{\&}\eta(\star,x)=\bigvee_{x\in X}\mu^*(x,\star)\mathbin{\&}\mu(\star,x)=(\mu^*\circ\mu)(\star,\star)=k,$$

and

$$\begin{aligned}
(\zeta\circ\eta)(a,a)&=\bigvee_{x\in X}\mu(\star,x\cdot a)\mathbin{\&}\mu^*(x\cdot a,\star)\\
&\leqslant\bigvee_{x\in X}\mathrm{id}_X(x\cdot a,x\cdot a) && (\mu\dashv\mu^*)\\
&=k\\
&=(\mu^*\circ\mu)(\star,\star) && (\text{Lemma 3.4})\\
&=\bigvee_{x\in X}\mu^*(x,\star)\mathbin{\&}\mu(\star,x)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{x\in X} \mu^*(x\cdot a^{-1}\cdot a,\star) \mathbin{\&} \mu(\star, x\cdot a^{-1}\cdot a)\\
&\leqslant \bigvee_{y\in X} \mu^*(y\cdot a,\star) \mathbin{\&} \mu(\star, y\cdot a)\\
&= (\zeta\circ\eta)(a,a)
\end{aligned}
$$

for all $a \in X \setminus \{e\}$, as desired.

Second, $\zeta\circ\mu = \tau_{X\setminus\{e\}}$ and $\eta\circ\tau_{X\setminus\{e\}} = \mu$, so that $\zeta\colon (X,\mu)\longrightarrow((X\setminus\{e\})_+,\tau_{X\setminus\{e\}})$ and $\eta\colon ((X\setminus\{e\})_+,\tau_{X\setminus\{e\}})\longrightarrow(X,\mu)$ are isomorphisms in $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$. Indeed, $\zeta\circ\mu = \tau_{X\setminus\{e\}}$ because

$$
\begin{aligned}
(\zeta\circ\mu)(\star,\star) &= \bigvee_{x\in X} \zeta(x,\star) \mathbin{\&} \mu(\star,x)\\
&= \bigvee_{x\in X} \mu^*(x,\star) \mathbin{\&} \mu(\star,x)\\
&= (\mu^*\circ\mu)(\star,\star)\\
&= k && \text{(Lemma 3.4)}\\
&= \tau_{X\setminus\{e\}}(\star,\star)
\end{aligned}
$$

and

$$
\begin{aligned}
(\zeta\circ\mu)(\star,a) &= \bigvee_{x\in X} \zeta(x,a) \mathbin{\&} \mu(\star,x)\\
&= \bigvee_{x\in X} \mu(\star,x) \mathbin{\&} \mu^*(x\cdot a,\star)\\
&= \bot && \text{((G2) and Lemma 3.4)}\\
&= \tau_{X\setminus\{e\}}(\star,a)
\end{aligned}
$$

for all $a \in X\setminus\{e\}$. Meanwhile, $\eta\circ\tau_{X\setminus\{e\}} = \mu$ because

$$
(\eta\circ\tau_{X\setminus\{e\}})(\star,x) = \bigvee_{a\in(X\setminus\{e\})_+} \eta(a,x) \mathbin{\&} \tau_{X\setminus\{e\}}(\star,a) = \eta(\star,x) \mathbin{\&} k = \eta(\star,x) = \mu(\star,x)
$$

for all $x\in X$, where the second equality follows from (4.iv). □

Lemma 4.9 guarantees that $\mathsf{K}\colon \mathsf{Q}\text{-}\mathbf{Map}^{\partial}\longrightarrow\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$ is an equivalence of categories:

**Theorem 4.10.** *Assuming the axiom of choice,* $\mathsf{Q}\text{-}\mathbf{Map}^{\partial}$ *is equivalent to* $\{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$ *and, therefore, is monadic over* $\mathsf{Q}\text{-}\mathbf{Map}$.

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