Quantales as formal concept lattices

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Abstract

We establish a new bridge connecting quantales, semigroups and the theory of formal concept analysis. By introducing residuated relations whose domains are semigroups, we show that every quantale arises as the formal concept lattice induced by such a relation. Furthermore, this construction yields an equivalence between the category of quantales and a category of such residuated relations whose morphisms are multiplicative bonds.

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1. Introduction

Quantales [27, 35] are complete lattices equipped with a multiplication distributing over arbitrary suprema. They provide a flexible ordered-algebraic framework in which completeness and associative multiplication interact, and have been studied from a variety of algebraic, geometric, and topological perspectives [32, 33, 34, 43, 42, 25]. *Formal concept analysis* (FCA), on the other hand, starts from a relation between an object set and an attribute set (called a *formal context*), and forms a complete lattice through its Galois closure (called the *formal concept lattice*) [14, 8]. The two constructions meet naturally only after one answers a structural question: if the object set is additionally equipped with a semigroup structure, when does the induced multiplication on its subsets descend through the Galois closure?

This paper answers that question at the level of relations. A *residuated relation*

$$R\colon X \longrightarrow\!\!\!\!\!\mapsto Y$$

with X a semigroup and Y a set allows an attribute of a product to be transported to a residual attribute of either factor (Definition 4.1). We prove that this elementary condition is strong enough to make the closure

$$\mathsf{j}_R = R^{\downarrow}R^{\uparrow}\colon \mathsf{P}X \longrightarrow \mathsf{P}X$$

a quantic nucleus. Consequently its fixed points, equivalently the extents of the context R, form a quantale. We then prove that every quantale arises in this way, and characterize the residuated relations presenting a given quantale Q (Theorem 4.4). Thus quantales admit relation-level presentations more flexible than the canonical order presentation: the object set carries a semigroup structure, while residual attributes record how its multiplication interacts with the incidence relation.

The object-level construction has a natural morphism-level lift. Ordinary *bonds* [13] between formal contexts form a category **Bond** equivalent to the category **Sup** of complete lattices and sup-preserving

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maps [26]; in the quantaloid setting this is the *back diagonal* construction of [38]. We show that, for residuated relations, the condition

$$\mathbf{b}[xx'] = \mathbf{b}[x] \otimes_S \mathbf{b}[x']$$

selects exactly those bonds whose induced maps preserve the descended multiplication. The resulting category

$$\mathbf{ResRel}_\otimes$$

of residuated relations and multiplicative bonds is equivalent to **Quant** (Theorem 6.5).

The general Galois and monoidal frameworks used here are already established: Resende's theory of sup-lattice 2-forms encompasses the Galois quotient associated with a formal context [31], while Mori established the lattice tensor on bonds and the resulting strong monoidal equivalence with **Sup**; see [26] and [41, Theorem 27]. The contribution of the present paper is the relation-level algebra added to these frameworks: the residual condition on a semigroup domain forces the Galois closure to be a quantic nucleus, and the multiplicative row condition characterizes exactly the bonds inducing quantale homomorphisms.

Moreover, Section 7 gives a categorical interpretation of the main theorem in terms of semigroup objects in **Bond**, while Section 8 gives a focused application: the enveloping quantal frame constructed in [32] from an abstract complete pseudogroup is obtained as a concept quantale.

2. Quantales

A *quantale* [35] is a complete lattice Q equipped with an associative binary operation $*\colon Q \times Q \longrightarrow Q$ such that the multiplication preserves arbitrary suprema in each variable:

$$\Big(\bigvee_{i\in I} p_i\Big) * q = \bigvee_{i\in I}(p_i * q) \quad \text{and} \quad p * \Big(\bigvee_{i\in I} q_i\Big) = \bigvee_{i\in I}(p * q_i) \tag{2.i}$$

for all $p, q \in Q$ and all families $\{p_i\}_{i\in I}$ and $\{q_i\}_{i\in I}$ in Q. A *homomorphism* of quantales is a map $f\colon P \longrightarrow Q$ which preserves arbitrary suprema and multiplication:

$$f\Big(\bigvee_{i\in I} p_i\Big) = \bigvee_{i\in I} f(p_i), \qquad f(p * p') = f(p) * f(p')$$

for all $p, p' \in P$ and all families $\{p_i\}_{i\in I}$ in P. We denote by

$$\mathbf{Quant}$$

the category of quantales and their homomorphisms.

Remark 2.1. Unless otherwise stated, quantales in this paper are not assumed to be unital. The unital version is obtained by requiring quantales to have a multiplicative unit and homomorphisms to preserve it.

Since the multiplication in a quantale preserves arbitrary suprema in each variable, for any $p, q \in Q$ the maps

$$- * q\colon Q \longrightarrow Q \quad \text{and} \quad p * -\colon Q \longrightarrow Q$$

both admit right adjoints. We write these right adjoints as

$$- \swarrow q\colon Q \longrightarrow Q \quad \text{and} \quad p \searrow -\colon Q \longrightarrow Q,$$

respectively. Thus,

$$p * q \leqslant r \iff p \leqslant r \swarrow q \iff q \leqslant p \searrow r \tag{2.ii}$$

for all $p, q, r \in Q$. Explicitly,

$$r \swarrow q = \bigvee\{p \in Q \mid p * q \leqslant r\}, \qquad p \searrow r = \bigvee\{q \in Q \mid p * q \leqslant r\}.$$

These residual operations will be used later to construct canonical examples of residuated relations.

We write the product of a *semigroup* [7, 17] X by juxtaposition: thus the product of $x, x' \in X$ is xx'. The symbol $*$ is reserved for the multiplication of a quantale. Every semigroup X gives rise to a quantale $\mathsf{P}X$, whose underlying complete lattice is the powerset of X and whose multiplication is defined by

$$A * B = \{ab \mid a \in A,\ b \in B\}$$

for all $A, B \subseteq X$. Thus $(\mathsf{P}X, *)$ is the *free quantale* generated by X:

Proposition 2.2. *The assignment $X \mapsto (\mathsf{P}X, *)$ defines a left adjoint to the forgetful functor*

$$\mathbf{Quant} \longrightarrow \mathbf{SGrp},$$

where **SGrp** *is the category of semigroups and their homomorphisms.*

This is the non-unital analogue of Rosenthal's free-quantale adjunction [35, Proposition 2.3.2]; the same proof applies with units omitted.

A *quantic nucleus* [35] on a quantale Q is a map $\mathsf{j}\colon Q \longrightarrow Q$ such that

(N1) $x \leqslant y \implies \mathsf{j}(x) \leqslant \mathsf{j}(y)$,

(N2) $x \leqslant \mathsf{j}(x)$,

(N3) $\mathsf{jj}(x) = \mathsf{j}(x)$,

(N4) $\mathsf{j}(x) * \mathsf{j}(y) \leqslant \mathsf{j}(x * y)$

for all $x, y \in Q$. Thus a quantic nucleus is a closure operator compatible with the quantale multiplication. As an immediate consequence of (N1)–(N4), we have

$$\mathsf{j}(x * y) \leqslant \mathsf{j}(\mathsf{j}(x) * \mathsf{j}(y)) \leqslant \mathsf{jj}(x * y) = \mathsf{j}(x * y),$$

and consequently

$$\mathsf{j}(x * y) = \mathsf{j}(\mathsf{j}(x) * \mathsf{j}(y)) \tag{2.iii}$$

for all $x, y \in Q$.

Each quantic nucleus j on Q induces a quotient quantale, denoted by Q_{j}. Its underlying set is the set of fixed points

$$Q_{\mathsf{j}} = \{x \in Q \mid \mathsf{j}(x) = x\}.$$

For $x, y, x_i \in Q_{\mathsf{j}}$ $(i \in I)$, the multiplication and suprema in Q_{j} are given by

$$x *_{\mathsf{j}} y = \mathsf{j}(x * y), \qquad \bigsqcup_{i \in I} x_i = \mathsf{j}\Big(\bigvee_{i \in I} x_i\Big), \tag{2.iv}$$

where $\bigsqcup$ denotes the supremum in Q_{j} and $\bigvee$ denotes the supremum in Q. Moreover, $\mathsf{j}\colon Q \longrightarrow Q_{\mathsf{j}}$ is a homomorphism of quantales.

The following standard representation theorem says that every quantale is a quotient of a free quantale by a quantic nucleus.

Theorem 2.3. *(See [35, Theorem 3.1.2].) For every quantale Q, there exist a semigroup X and a quantic nucleus j on $\mathsf{P}X$ such that*

$$Q \cong (\mathsf{P}X)_{\mathsf{j}}$$

as quantales.

Remark 2.4. The preceding theorem will be refined in this paper. We shall show that the nuclei relevant for representing quantales can be obtained from Galois closures induced by residuated relations. More precisely, a residuated relation $R\colon X \longrightarrow\!\!\!\!\!+ Y$ will induce the quantic nucleus

$$\mathsf{j}_R := R^{\downarrow} R^{\uparrow}\colon \mathsf{P}X \longrightarrow \mathsf{P}X,$$

and the corresponding quotient quantale will be precisely the formal concept lattice of R.

3. Formal concept lattices

Recall that a *relation* $R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ between sets is a subset $R \subseteq X \times Y$, where $(x, y) \in R$ is also written as xRy. Each relation $R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ induces two maps

$$R^{\uparrow}\colon \mathsf{P}X \longrightarrow \mathsf{P}Y \quad \text{and} \quad R^{\downarrow}\colon \mathsf{P}Y \longrightarrow \mathsf{P}X$$

between the powersets of X and Y, defined by

$$R^{\uparrow}(A) = \{y \in Y \mid \forall a \in A,\ aRy\}, \qquad R^{\downarrow}(B) = \{x \in X \mid \forall b \in B,\ xRb\}. \tag{3.i}$$

Then $R^{\uparrow}$ and $R^{\downarrow}$ form a Galois connection between $\mathsf{P}X$ and $(\mathsf{P}Y)^{\mathrm{op}}$:

$$B \subseteq R^{\uparrow}(A) \iff A \subseteq R^{\downarrow}(B) \tag{3.ii}$$

for all $A \subseteq X$ and $B \subseteq Y$. Consequently,

- both $R^{\uparrow}$ and $R^{\downarrow}$ send unions into intersections, and
- both $R^{\downarrow}R^{\uparrow}\colon \mathsf{P}X \longrightarrow \mathsf{P}X$ and $R^{\uparrow}R^{\downarrow}\colon \mathsf{P}Y \longrightarrow \mathsf{P}Y$ are closure operators.

Remark 3.1. The Galois connection above is standard in formal concept analysis. It also admits a formulation in the theory of sup-lattice 2-forms: the relation R determines a 2-form on the powersets of X and Y, and the associated orthogonality construction recovers the same Galois correspondence and concept lattice; see [31]. We use the ordinary FCA notation throughout.

Remark 3.2. More generally, every distributor between enriched categories induces an *Isbell adjunction*. If sets X and Y are regarded as discrete categories enriched in the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$, then R is a $\mathbf{2}$-distributor between them, and the Isbell adjunction induced by R specializes to the Galois connection $R^{\uparrow} \dashv R^{\downarrow}$ between $\mathsf{P}X$ and $(\mathsf{P}Y)^{\mathrm{op}}$; see [23, 9, 19, 40, 39, 37, 21].

For $x \in X$ and $y \in Y$, write

$$R[x] = \{y' \in Y \mid xRy'\} \qquad \text{and} \qquad R^{-1}[y] = \{x' \in X \mid x'Ry\}$$

for the row of x and the column of y, respectively. Thus

$$R^{\uparrow}(\{x\}) = R[x], \qquad R^{\downarrow}(\{y\}) = R^{-1}[y].$$

More generally,

$$R^{\uparrow}(A) = \bigcap_{x \in A} R[x], \qquad R^{\downarrow}(B) = \bigcap_{y \in B} R^{-1}[y]$$

for all $A \subseteq X$ and $B \subseteq Y$.

Lemma 3.3. *Let $R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ be a relation between sets. For each $A \subseteq X$ and $x \in X$, the following statements are equivalent:*

(i) $x \in R^{\downarrow}R^{\uparrow}(A)$.

(ii) *For each $y \in Y$, if aRy for all $a \in A$, then xRy.*

Proof. By definition, $x \in R^{\downarrow}R^{\uparrow}(A)$ if and only if xRy for every $y \in R^{\uparrow}(A)$. Since $y \in R^{\uparrow}(A)$ precisely means that aRy for all $a \in A$, the assertion follows. $\square$

In the theory of *formal concept analysis* [14, 8], a *formal context* (or *context* for short) is a triple (X, Y, R), where X is the set of *objects*, Y is the set of *attributes*, and $R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ is a relation. The statement xRy is read as "the object x has the attribute y".

A *formal concept* of a formal context (X, Y, R) is a pair (A, B), where $A \subseteq X$ and $B \subseteq Y$, such that

$$A = R^{\downarrow}(B) \quad \text{and} \quad B = R^{\uparrow}(A).$$

In this case, A is called the *extent* of the concept and B is called its *intent*. The set of all formal concepts is denoted by $\mathfrak{B}(X, Y, R)$ (or $\mathfrak{B}R$ for short), which is ordered by

$$(A, B) \leqslant (A', B') \iff A \subseteq A' \iff B \supseteq B'.$$

With this order, $\mathfrak{B}R$ is a complete lattice, called the *formal concept lattice* (or *concept lattice* for short) of R.

A subset $A \subseteq X$ is called an R-*extent* if $A = R^{\downarrow}R^{\uparrow}(A)$, and a subset $B \subseteq Y$ is called an R-*intent* if $B = R^{\uparrow}R^{\downarrow}(B)$. We write

$$\begin{aligned}\mathrm{Ext}(R) &= \mathsf{Fix}(R^{\downarrow}R^{\uparrow}) = \{A \subseteq X \mid A = R^{\downarrow}R^{\uparrow}(A)\} \quad \text{and}\\ \mathrm{Int}(R) &= \mathsf{Fix}(R^{\uparrow}R^{\downarrow}) = \{B \subseteq Y \mid B = R^{\uparrow}R^{\downarrow}(B)\}.\end{aligned}$$

The assignments

$$(A, B) \mapsto A \quad \text{and} \quad (A, B) \mapsto B$$

identify $\mathfrak{B}R$ with $\mathrm{Ext}(R)$ and with $\mathrm{Int}(R)^{\mathrm{op}}$, respectively. In what follows, we shall mostly use the extent representation and write

$$\mathsf{M}R = \mathrm{Ext}(R) = \mathsf{Fix}(R^{\downarrow}R^{\uparrow}).$$

Thus $\mathsf{M}R$ is a complete lattice whose order is inclusion. Its infima are computed as intersections:

$$\bigsqcap_{i \in I} A_i = \bigcap_{i \in I} A_i,$$

and its suprema are computed by closing unions:

$$\bigsqcup_{i \in I} A_i = R^{\downarrow}R^{\uparrow}\Big(\bigcup_{i \in I} A_i\Big),$$

where $A_i \in \mathsf{M}R$ for all $i \in I$. Similarly, $\mathrm{Int}(R)$, ordered by inclusion, is a complete lattice: its infima are intersections, and its suprema are obtained by closing unions under $R^{\uparrow}R^{\downarrow}$. Moreover, the restrictions

$$R^{\uparrow}\big|_{\mathsf{M}R} : \mathsf{M}R \longrightarrow \mathrm{Int}(R)^{\mathrm{op}} \quad \text{and} \quad R^{\downarrow}\big|_{\mathrm{Int}(R)} : \mathrm{Int}(R)^{\mathrm{op}} \longrightarrow \mathsf{M}R \tag{3.iii}$$

are mutually inverse order isomorphisms.

We shall also need the standard notion of a bond between formal contexts [13]. Let

$$R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y \quad \text{and} \quad S\colon Z \longrightarrow\!\!\!\!\!\!\mapsto W$$

be two relations. A *bond* from R to S is a relation $\mathbf{b}\colon X \longrightarrow\!\!\!\!\!\!\mapsto W$ such that

(B1) for every $x \in X$, the row $\mathbf{b}[x] = \{w \in W \mid x\mathbf{b}w\}$ is an S-intent;

(B2) for every $w \in W$, the column $\mathbf{b}^{-1}[w] = \{x \in X \mid x\mathbf{b}w\}$ is an R-extent.

Here rows are subsets of the attribute set W of the second context, while columns are subsets of the object set X of the first context.

Every bond $\mathbf{b}\colon R \longrightarrow S$ induces a map

$$\mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}S, \quad \mathbf{b}_*(A) = S^{\downarrow}\mathbf{b}^{\uparrow}(A), \tag{3.iv}$$

where $\mathbf{b}^{\uparrow}$ is defined by (3.i), i.e.,

$$\mathbf{b}^{\uparrow}(A) = \{w \in W \mid \forall a \in A,\ a\mathbf{b}w\}.$$

Since $\mathbf{b}^{\uparrow}(A) \subseteq W$, the set $S^{\downarrow}\mathbf{b}^{\uparrow}(A)$ is an S-extent, and therefore $\mathbf{b}_*(A) \in \mathsf{M}S$.

The following standard fact is the well-known correspondence between bonds of formal contexts and sup-preserving maps between their concept lattices; see, for example, [13, 26]. It is also the relational instance of the quantaloid-theoretic bond construction in [38]. We recall its elementary verification in the present notation.

Proposition 3.4. *For every bond $\mathbf{b}\colon R \longrightarrow S$, the induced map $\mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}S$ preserves arbitrary suprema.*

Proof. Let $A_i \in \mathsf{M}R$ for $i \in I$. Since $\mathbf{b}^{\uparrow}$ sends unions to intersections, we have

$$\mathbf{b}^{\uparrow}\Big(\bigcup_{i\in I} A_i\Big) = \bigcap_{i\in I} \mathbf{b}^{\uparrow}(A_i).$$

Moreover, the columns of $\mathbf{b}$ are R-extents, hence replacing a subset $A \subseteq X$ by its closure $R^{\downarrow}R^{\uparrow}(A)$ does not change $\mathbf{b}^{\uparrow}(A)$. Indeed, for every $w \in W$,

$$w \in \mathbf{b}^{\uparrow}(A) \iff A \subseteq \mathbf{b}^{-1}[w],$$

and $\mathbf{b}^{-1}[w]$ is an R-extent.

Therefore,

$$\mathbf{b}_*\Big(\bigsqcup_{i\in I} A_i\Big) = S^{\downarrow}\mathbf{b}^{\uparrow}R^{\downarrow}R^{\uparrow}\Big(\bigcup_{i\in I} A_i\Big) = S^{\downarrow}\mathbf{b}^{\uparrow}\Big(\bigcup_{i\in I} A_i\Big) = S^{\downarrow}\Big(\bigcap_{i\in I} \mathbf{b}^{\uparrow}(A_i)\Big).$$

Finally, each $\mathbf{b}^{\uparrow}(A_i)$ is an S-intent because $\mathbf{b}$ has S-intent rows and intents are closed under arbitrary intersections. Hence $S^{\downarrow}$ sends the above intersection of intents to the supremum of the corresponding S-extents in $\mathsf{M}S$ (see (3.iii)), giving

$$S^{\downarrow}\Big(\bigcap_{i\in I} \mathbf{b}^{\uparrow}(A_i)\Big) = \bigsqcup_{i\in I} S^{\downarrow}\mathbf{b}^{\uparrow}(A_i) = \bigsqcup_{i\in I} \mathbf{b}_*(A_i).$$

Thus $\mathbf{b}_*$ preserves arbitrary suprema. $\square$

Conversely, every sup-preserving map between concept lattices is induced by a unique bond. More precisely, let $h\colon \mathsf{M}R \longrightarrow \mathsf{M}S$ be a sup-preserving map. Define a relation $\mathbf{b}_h\colon X \longrightarrow\!\!\!\!\!\!+ \; W$ by (cf. (3.ii))

$$x\mathbf{b}_h w \iff w \in S^{\uparrow}hR^{\downarrow}R^{\uparrow}(\{x\}) \iff hR^{\downarrow}R^{\uparrow}(\{x\}) \subseteq S^{\downarrow}(\{w\}). \tag{3.v}$$

Proposition 3.5. *Let $h\colon \mathsf{M}R \longrightarrow \mathsf{M}S$ be a sup-preserving map. Then $\mathbf{b}_h$ is a bond from R to S, and $(\mathbf{b}_h)_* = h$.*

Proof. First, for each $w \in W$, the column $\mathbf{b}_h^{-1}[w]$ is given by

$$\mathbf{b}_h^{-1}[w] = \{x \in X \mid hR^{\downarrow}R^{\uparrow}(\{x\}) \subseteq S^{\downarrow}(\{w\})\}.$$

Since h preserves arbitrary suprema, this set is an R-extent. Indeed, if $A = R^{\downarrow}R^{\uparrow}(\mathbf{b}_h^{-1}[w])$, then

$$h(A) = h\Big(\bigsqcup_{x\in\mathbf{b}_h^{-1}[w]} R^{\downarrow}R^{\uparrow}(\{x\})\Big) = \bigsqcup_{x\in\mathbf{b}_h^{-1}[w]} hR^{\downarrow}R^{\uparrow}(\{x\}) \subseteq S^{\downarrow}(\{w\}).$$

If $x \in A$, then $R^{\downarrow}R^{\uparrow}(\{x\}) \subseteq A$ (because A is an R-extent). As every sup-preserving map is monotone, it follows that

$$hR^{\downarrow}R^{\uparrow}(\{x\}) \subseteq h(A) \subseteq S^{\downarrow}(\{w\}).$$

Hence $x \in \mathbf{b}_h^{-1}[w]$. Therefore $A \subseteq \mathbf{b}_h^{-1}[w]$, and so $\mathbf{b}_h^{-1}[w]$ is an R-extent.

Next, for each $x \in X$, the row $\mathbf{b}_h[x]$ is an S-intent because

$$\mathbf{b}_h[x] = S^{\uparrow}hR^{\downarrow}R^{\uparrow}(\{x\}).$$

Thus $\mathbf{b}_h$ is a bond.

Finally, for every $A \in \mathsf{M}R$,

$$\begin{aligned}
(\mathbf{b}_h)_*(A) &= S^{\downarrow}\mathbf{b}_h^{\uparrow}(A) \\
&= S^{\downarrow}(\{w \in W \mid \forall a \in A,\ a\mathbf{b}_h w\}) \\
&= S^{\downarrow}\Big(\bigcap_{a\in A} S^{\uparrow}hR^{\downarrow}R^{\uparrow}(\{a\})\Big) \\
&= S^{\downarrow}S^{\uparrow}\Big(\bigsqcup_{a\in A} hR^{\downarrow}R^{\uparrow}(\{a\})\Big) \\
&= \bigsqcup_{a\in A} hR^{\downarrow}R^{\uparrow}(\{a\}) \\
&= h\Big(\bigsqcup_{a\in A} R^{\downarrow}R^{\uparrow}(\{a\})\Big) \\
&= h(A).
\end{aligned}$$

This proves $(\mathbf{b}_h)_* = h$. □

Corollary 3.6. *For relations $R\colon X \looparrowright Y$ and $S\colon Z \looparrowright W$, bonds from R to S are in one-to-one correspondence with sup-preserving maps $\mathsf{M}R \longrightarrow \mathsf{M}S$.*

Proof. By Propositions 3.4 and 3.5, each bond $\mathbf{b}$ induces a sup-preserving map $\mathbf{b}_*$, and each sup-preserving map h induces a bond $\mathbf{b}_h$ with $(\mathbf{b}_h)_* = h$.

It remains only to observe uniqueness. Let $\mathbf{b}$ be a bond, and fix $x \in X$ and $w \in W$. Put

$$C_x = R^{\downarrow}R^{\uparrow}(\{x\}).$$

Since $\mathbf{b}^{-1}[w]$ is an R-extent and C_x is the least R-extent containing x, we have

$$x\mathbf{b}w \iff C_x \subseteq \mathbf{b}^{-1}[w] \iff w \in \mathbf{b}^{\uparrow}(C_x).$$

Moreover, $\mathbf{b}^{\uparrow}(C_x)$ is an S-intent, being the intersection of the S-intent rows $\mathbf{b}[a]$ for $a \in C_x$. Hence

$$w \in \mathbf{b}^{\uparrow}(C_x) \iff w \in S^{\uparrow}S^{\downarrow}\mathbf{b}^{\uparrow}(C_x) \iff S^{\downarrow}\mathbf{b}^{\uparrow}(C_x) \subseteq S^{\downarrow}(\{w\}).$$

By the definition of $\mathbf{b}_*$, this gives

$$x\mathbf{b}w \iff \mathbf{b}_*R^{\downarrow}R^{\uparrow}(\{x\}) \subseteq S^{\downarrow}(\{w\}).$$

Hence $\mathbf{b}$ is recovered from $\mathbf{b}_*$ by formula (3.v). □

The preceding correspondence also supplies the standard category of bonds. Let

Bond

have formal contexts as objects and bonds as morphisms. The identity morphism on a context

$$R\colon X \longrightarrow\!\!\!\!\!\!\rightarrow Y$$

is the relation R itself: its rows are the R-intents $R^{\uparrow}(\{x\})$, its columns are the R-extents $R^{\downarrow}(\{y\})$, and

$$R_* = 1_{\mathsf{M}R}.$$

For bonds

$$\mathbf{b}\colon R \longrightarrow S \qquad \text{and} \qquad \mathbf{c}\colon S \longrightarrow T,$$

where

$$R\colon X \longrightarrow\!\!\!\!\!\!\rightarrow Y, \qquad S\colon Z \longrightarrow\!\!\!\!\!\!\rightarrow W, \qquad T\colon U \longrightarrow\!\!\!\!\!\!\rightarrow V,$$

define their composite $\mathbf{c} \circ \mathbf{b}\colon R \longrightarrow T$ to be the unique bond corresponding under Corollary 3.6 to the sup-preserving map

$$\mathbf{c}_* \circ \mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}T.$$

Since $\mathbf{b}^{\uparrow}R^{\downarrow}R^{\uparrow}(\{x\}) = \mathbf{b}[x]$, one has

$$\mathbf{b}_*R^{\downarrow}R^{\uparrow}(\{x\}) = S^{\downarrow}(\mathbf{b}[x]).$$

Moreover, the recovery formula established in the proof of Corollary 3.6, together with preservation of suprema, gives

$$\mathbf{c}_*(A) \subseteq T^{\downarrow}(\{v\}) \iff A \subseteq \mathbf{c}^{-1}[v]$$

for every $A \in \mathsf{M}S$ and $v \in V$. It follows that the composite has the elementwise description

$$x(\mathbf{c} \circ \mathbf{b})v \iff S^{\downarrow}(\mathbf{b}[x]) \subseteq \mathbf{c}^{-1}[v]$$

for all $x \in X$ and $v \in V$, and by construction

$$(\mathbf{c} \circ \mathbf{b})_* = \mathbf{c}_* \circ \mathbf{b}_*.$$

Since bonds are uniquely determined by their induced maps, associativity and the unit laws follow from the corresponding laws for composition of maps. Thus **Bond** is a category.

Remark 3.7. The category **Bond** has a quantaloid-theoretic interpretation. The notion of a bond above is the specialization to the quantaloid **Rel** (see [36] for the theory of *quantaloids*) of the notion of a bond between arrows in an arbitrary quantaloid [38]. More generally, one begins with Chu connections and forms their quotient quantaloid of back diagonals. For **Rel**, the resulting quantaloid is denoted by $\mathbf{B}(\mathbf{Rel})$, and it is canonically isomorphic to **Bond** (see [38, Proposition 2.3.5]). Under this identification, a back diagonal from $R\colon X \longrightarrow\!\!\!\!\!\!\rightarrow Y$ to $S\colon Z \longrightarrow\!\!\!\!\!\!\rightarrow W$ is exactly a relation $\mathbf{b}\colon X \longrightarrow\!\!\!\!\!\!\rightarrow W$ whose rows are S-intents and whose columns are R-extents. Thus the present definition is the elementwise form of the back-diagonal construction.

Consider the concept-lattice assignment

$$\mathsf{M}_0\colon \mathbf{Bond} \longrightarrow \mathbf{Sup}, \qquad R \mapsto \mathsf{M}R, \qquad \mathbf{b} \mapsto \mathbf{b}_*.$$

Corollary 3.6 says that this functor is fully faithful. The basic theorem on concept lattices (see [14, Theorem 3] and [8, Theorem 3.9]) says that every complete lattice is isomorphic to the concept lattice of a formal context, so M_0 is essentially surjective. Hence it is an equivalence of categories:

Proposition 3.8. *(See [26, 38].) The functor* $\mathsf{M}_0\colon \mathbf{Bond} \longrightarrow \mathbf{Sup}$ *is an equivalence of categories.*

More precisely, this is the bond formulation of Mori's Galois equivalence [26, Theorem 73]; in the quantaloid setting, it is the case $\mathcal{Q} = \mathbf{2}$ of [38, Proposition 2.3.5 and Theorem 3.4.4].

4. Quantales as formal concept lattices

We now explain how quantales arise as formal concept lattices of residuated relations. The key point is that, for a semigroup X and a set Y, a relation

$$R \colon X \rightarrow\!\!\!\!\!\!+ \; Y$$

induces a Galois closure

$$\mathsf{j}_R := R^{\downarrow}R^{\uparrow} \colon \mathsf{P}X \longrightarrow \mathsf{P}X,$$

and, under a residuation condition, this closure is a quantic nucleus on the free quantale $\mathsf{P}X$.

Definition 4.1. Let X be a semigroup and Y a set. A relation $R\colon X \rightarrow\!\!\!\!\!\!+ \; Y$ is *residuated* if there exist maps

$$\swarrow \colon Y \times X \longrightarrow Y \quad \text{and} \quad \searrow \colon X \times Y \longrightarrow Y$$

such that

$$(xx', y) \in R \iff (x, y \swarrow x') \in R \iff (x', x \searrow y) \in R \tag{4.i}$$

for all $x, x' \in X$ and $y \in Y$.

Equivalently, writing xRy for $(x, y) \in R$, the condition says that the semigroup multiplication on the domain X admits right residuals with respect to the relation R:

$$(xx')Ry \iff xR(y \swarrow x') \iff x'R(x \searrow y).$$

Thus $y \swarrow x'$ and $x \searrow y$ play the role of the right residuals of y by x' and of y by x, respectively, but only relative to the relation R.

A basic special case is obtained from ordered semigroups. By a *residuated ordered semigroup* we mean an ordered semigroup X whose order relation

$$\leqslant \colon X \rightarrow\!\!\!\!\!\!+ \; X$$

is a residuated relation. Explicitly, this means that there exist maps

$$/ \colon X \times X \longrightarrow X \quad \text{and} \quad \backslash \colon X \times X \longrightarrow X$$

such that

$$xx' \leqslant y \iff x \leqslant y/x' \iff x' \leqslant x \backslash y$$

for all $x, x', y \in X$. In this case, the residuals are internal to the ordered semigroup X.

In particular, as an immediate consequence of the residual adjunctions (2.ii), every quantale Q is a residuated ordered semigroup. Example A.1 extends this basic case from Q itself to sup-dense and inf-dense residuated ordered subsemigroups $A \subseteq Q$. Further concrete realizations of residuated relations are collected in Appendix A.

Proposition 4.2. *Let $R\colon X \rightarrow\!\!\!\!\!\!+ \; Y$ be a residuated relation. Then j_R is a quantic nucleus on the quantale $\mathsf{P}X$.*

Proof. The map $\mathsf{j}_R = R^{\downarrow}R^{\uparrow} \colon \mathsf{P}X \longrightarrow \mathsf{P}X$ is already a closure operator, since it is induced by the Galois connection between $\mathsf{P}X$ and $(\mathsf{P}Y)^{\mathrm{op}}$. It remains to show that it is compatible with the quantale multiplication on $\mathsf{P}X$ (see (N4)); that is,

$$(R^{\downarrow}R^{\uparrow}(A)) * (R^{\downarrow}R^{\uparrow}(B)) \subseteq R^{\downarrow}R^{\uparrow}(A * B)$$

for all $A, B \subseteq X$.

Let $x \in R^{\downarrow}R^{\uparrow}(A)$ and $x' \in R^{\downarrow}R^{\uparrow}(B)$. We show that $xx' \in R^{\downarrow}R^{\uparrow}(A * B)$. By Lemma 3.3, it suffices to prove that, for each $y \in Y$, if $(ab)Ry$ for all $a \in A$ and $b \in B$, then $(xx')Ry$. Suppose that $(ab)Ry$ for all $a \in A$ and $b \in B$. For every $b \in B$, residuation gives

$$(ab)Ry \iff aR(y \swarrow b).$$

Hence $aR(y \swarrow b)$ for all $a \in A$. Since $x \in R^{\downarrow}R^{\uparrow}(A)$, Lemma 3.3 gives $xR(y \swarrow b)$. Again by residuation, this is equivalent to $bR(x \searrow y)$. Thus $bR(x \searrow y)$ for every $b \in B$. Since $x' \in R^{\downarrow}R^{\uparrow}(B)$, Lemma 3.3 gives $x'R(x \searrow y)$. By residuation once more, this is equivalent to $(xx')Ry$. Therefore $xx' \in R^{\downarrow}R^{\uparrow}(A * B)$, as required. □

Since j_R is a quantic nucleus on $\mathsf{P}X$, the quotient-quantale construction recalled in Section 2 applies. Its fixed-point lattice is precisely the lattice of R-extents:

$$(\mathsf{P}X)_{\mathsf{j}_R} = \mathsf{Fix}(\mathsf{j}_R) = \mathsf{M}R.$$

Thus the formulas in (2.iv) immediately yield the following canonical quantale structure on the concept lattice of R.

Proposition 4.3. *Let $R\colon X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$ be a residuated relation. Then $\mathsf{M}R$ is a quantale, with multiplication and suprema given by*

$$A \otimes_R B = R^{\downarrow}R^{\uparrow}(A * B), \qquad \bigsqcup_{i\in I} A_i = R^{\downarrow}R^{\uparrow}\Big(\bigcup_{i\in I} A_i\Big)$$

for all $A, B \in \mathsf{M}R$ and all families $\{A_i\}_{i\in I}$ in $\mathsf{M}R$. Moreover, the closure map

$$\mathsf{j}_R = R^{\downarrow}R^{\uparrow}\colon \mathsf{P}X \longrightarrow \mathsf{M}R$$

is a homomorphism of quantales. When the relation R is clear from the context, we simply write $A \otimes B$ instead of $A \otimes_R B$.

We now obtain the object-level representation theorem. Proposition 4.3 shows that every residuated relation $R\colon X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$ gives rise to a quantale $\mathsf{M}R$. The following theorem proves the converse: every quantale is isomorphic to $\mathsf{M}R$ for some residuated relation $R\colon X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$. More precisely, it characterizes the residuated relations presenting a given quantale. It is a quantale-theoretic refinement of the basic theorem on concept lattices (see [14, Theorem 3] and [8, Theorem 3.9]).

Let Q be a complete lattice. A map $f\colon X \longrightarrow Q$ is called *sup-dense* if

$$q = \bigvee\{f(x) \mid x \in X,\ f(x) \leqslant q\} \tag{4.ii}$$

for all $q \in Q$, and it is called *inf-dense* if

$$q = \bigwedge\{f(x) \mid x \in X,\ q \leqslant f(x)\} \tag{4.iii}$$

for all $q \in Q$.

Theorem 4.4. *Let Q be a quantale, and let $R\colon X \longrightarrow\!\!\!\!\!\!\shortmid\;\; Y$ be a residuated relation. Then Q is isomorphic to $\mathsf{M}R$ as a quantale if and only if there exist a sup-dense semigroup homomorphism $f\colon X \longrightarrow Q$ and an inf-dense map $g\colon Y \longrightarrow Q$ such that*

$$xRy \iff f(x) \leqslant g(y) \tag{4.iv}$$

for all $x \in X$ and $y \in Y$. In particular, every quantale Q is isomorphic, as a quantale, to $\mathsf{M}(Q, Q, \leqslant)$.

Proof. **Sufficiency.** Suppose first that such maps f and g exist. Define a map

$$\Phi\colon \mathsf{P}X \longrightarrow Q, \quad \Phi(A) = \bigvee f[A] = \bigvee\{f(a) \mid a \in A\}.$$

We show that the restriction of Φ to $\mathsf{M}R$ is an isomorphism of quantales.

First, $\Phi|_{\mathsf{M}R}\colon \mathsf{M}R \longrightarrow Q$ is injective. For $A \subseteq X$ and $y \in Y$, condition (4.iv) gives

$$\begin{aligned} y \in R^{\uparrow}(A) &\iff aRy \text{ for all } a \in A \\ &\iff f(a) \leqslant g(y) \text{ for all } a \in A \end{aligned}$$

$$\iff \bigvee f[A] \leqslant g(y).$$

Hence

$$R^{\uparrow}(A) = \{y \in Y \mid \Phi(A) \leqslant g(y)\}. \tag{4.v}$$

For $A \in \mathsf{M}R$, we claim that

$$A = \{x \in X \mid f(x) \leqslant \Phi(A)\},$$

hence an extent A is completely determined by $\Phi(A)$, which necessarily forces $\Phi|_{\mathsf{M}R}$ to be injective. Indeed, if $x \in A$, then clearly $f(x) \leqslant \Phi(A)$. Conversely, suppose that $f(x) \leqslant \Phi(A)$. To show $x \in A$, since $A = R^{\downarrow}R^{\uparrow}(A)$, it is enough to show that xRy for every $y \in R^{\uparrow}(A)$. But for every $y \in R^{\uparrow}(A)$, we have $\Phi(A) \leqslant g(y)$ by (4.v), and consequently

$$f(x) \leqslant \Phi(A) \leqslant g(y).$$

Thus xRy, and therefore $x \in A$.

Second, $\Phi|_{\mathsf{M}R} : \mathsf{M}R \longrightarrow Q$ is surjective. For every $q \in Q$, define

$$A_q = \{x \in X \mid f(x) \leqslant q\}.$$

We show that A_q is an R-extent. Since g is inf-dense, combining (4.iii) and (4.iv) we have

$$\begin{aligned} A_q &= \{x \in X \mid f(x) \leqslant g(y) \text{ for every } y \in Y \text{ with } q \leqslant g(y)\} \\ &= \{x \in X \mid xRy \text{ for every } y \in Y \text{ with } q \leqslant g(y)\} \\ &= R^{\downarrow}(\{y \in Y \mid q \leqslant g(y)\}). \end{aligned} \tag{4.vi}$$

Thus A_q is an R-extent. Moreover, since f is sup-dense, it follows from (4.ii) that

$$\Phi(A_q) = \bigvee\{f(x) \mid f(x) \leqslant q\} = q. \tag{4.vii}$$

Therefore, the restriction of Φ to $\mathsf{M}R$ is surjective.

Third, $\Phi|_{\mathsf{M}R} : \mathsf{M}R \longrightarrow Q$ preserves arbitrary suprema. Note that for every $U \subseteq X$, from (4.v), (4.vi) and (4.vii) we obtain

$$\Phi R^{\downarrow}R^{\uparrow}(U) = \Phi R^{\downarrow}(\{y \in Y \mid \Phi(U) \leqslant g(y)\}) = \Phi(A_{\Phi(U)}) = \Phi(U). \tag{4.viii}$$

Thus, for any family $\{A_i\}_{i\in I}$ in $\mathsf{M}R$,

$$\begin{aligned} \Phi\Big(\bigsqcup_{i\in I} A_i\Big) &= \Phi R^{\downarrow}R^{\uparrow}\Big(\bigcup_{i\in I} A_i\Big) \\ &= \Phi\Big(\bigcup_{i\in I} A_i\Big) \\ &= \bigvee f\Big[\bigcup_{i\in I} A_i\Big] \\ &= \bigvee_{i\in I}\bigvee f[A_i] \\ &= \bigvee_{i\in I} \Phi(A_i). \end{aligned}$$

Finally, $\Phi|_{\mathsf{M}R} : \mathsf{M}R \longrightarrow Q$ preserves multiplication. Since f is a semigroup homomorphism, we have

$$\begin{aligned} \Phi(A \otimes_R B) &= \Phi R^{\downarrow}R^{\uparrow}(A * B) \\ &= \Phi(A * B) && \text{(by (4.viii))} \end{aligned}$$

$$
\begin{aligned}
&= \bigvee f[A * B] \\
&= \bigvee \{f(ab) \mid a \in A,\ b \in B\} \\
&= \bigvee \{f(a) * f(b) \mid a \in A,\ b \in B\} \\
&= \Big(\bigvee f[A]\Big) * \Big(\bigvee f[B]\Big) && \text{(by (2.i))} \\
&= \Phi(A) * \Phi(B)
\end{aligned}
$$

for all $A, B \in \mathsf{M}R$, as desired.

Necessity. Suppose that

$$\Psi \colon \mathsf{M}R \longrightarrow Q$$

is an isomorphism of quantales. Define maps

$$f \colon X \longrightarrow Q, \quad f(x) = \Psi R^{\downarrow} R^{\uparrow}(\{x\}) \quad \text{and} \quad g \colon Y \longrightarrow Q, \quad g(y) = \Psi R^{\downarrow}(\{y\}).$$

Then f is a semigroup homomorphism, because

$$
\begin{aligned}
f(xx') &= \Psi R^{\downarrow} R^{\uparrow}(\{xx'\}) \\
&= \Psi R^{\downarrow} R^{\uparrow}(\{x\} * \{x'\}) \\
&= \Psi R^{\downarrow} R^{\uparrow}(R^{\downarrow} R^{\uparrow}(\{x\}) * R^{\downarrow} R^{\uparrow}(\{x'\})) && \text{(by (2.iii) and Proposition 4.2)} \\
&= \Psi(R^{\downarrow} R^{\uparrow}(\{x\}) \otimes_R R^{\downarrow} R^{\uparrow}(\{x'\})) \\
&= \Psi R^{\downarrow} R^{\uparrow}(\{x\}) * \Psi R^{\downarrow} R^{\uparrow}(\{x'\}) \\
&= f(x) * f(x')
\end{aligned}
$$

for all $x, x' \in X$.

For every extent $A \in \mathsf{M}R$ and every $x \in X$, one has

$$f(x) \leqslant \Psi(A) \iff R^{\downarrow} R^{\uparrow}(\{x\}) \subseteq A \iff x \in A.$$

Since

$$A = R^{\downarrow} R^{\uparrow}\Big(\bigcup_{x \in A} R^{\downarrow} R^{\uparrow}(\{x\})\Big) = \bigsqcup_{x \in A} R^{\downarrow} R^{\uparrow}(\{x\}),$$

it follows that

$$\Psi(A) = \bigvee_{x \in A} f(x) = \bigvee \{f(x) \mid f(x) \leqslant \Psi(A)\}.$$

Thus f is sup-dense.

Similarly, for every extent $A \in \mathsf{M}R$ and every $y \in Y$, one has

$$\Psi(A) \leqslant g(y) \iff A \subseteq R^{\downarrow}(\{y\}) \iff y \in R^{\uparrow}(A).$$

Since (cf. Lemma 3.3)

$$A = \bigcap_{y \in R^{\uparrow}(A)} R^{\downarrow}(\{y\}),$$

it follows that

$$\Psi(A) = \bigwedge_{y \in R^{\uparrow}(A)} g(y) = \bigwedge \{g(y) \mid \Psi(A) \leqslant g(y)\}.$$

Thus g is inf-dense.

Finally, for $x \in X$ and $y \in Y$,

$$xRy \iff x \in R^{\downarrow}(\{y\})$$

$$\begin{aligned}
&\iff R^{\downarrow}R^{\uparrow}(\{x\}) \subseteq R^{\downarrow}(\{y\}) \\
&\iff \Psi R^{\downarrow}R^{\uparrow}(\{x\}) \leqslant \Psi R^{\downarrow}(\{y\}) \\
&\iff f(x) \leqslant g(y).
\end{aligned}$$

This proves the converse implication. For the final assertion, take $X = Y = Q$, $R = \leqslant$, and $f = g = 1_Q$. As observed above, the order relation $\leqslant\colon Q \longrightarrow\!\!\!\!\!\!+\; Q$ is residuated, and 1_Q is both sup-dense and inf-dense. Hence $Q \cong \mathsf{M}(Q, Q, \leqslant)$ as quantales. □

Remark 4.5. The map g in Theorem 4.4 is not required to preserve any algebraic structure; indeed, the attribute set Y in Definition 4.1 is only a set. The role of g is to express the incidence relation by

$$xRy \iff f(x) \leqslant g(y)$$

and to be inf-dense. By contrast, the homomorphism property of f is used to show that $\Phi(A) = \bigvee f[A]$ preserves the induced quantale multiplication. In the canonical order presentation, both f and g are the identity map 1_Q.

Remark 4.6. Theorem 4.4 and the representation theorem of Brown and Gurr [5] both give relational representations of arbitrary quantales, but the relations play different roles. Brown and Gurr realize a quantale as a family of binary relations on a set, with multiplication given by relational composition. Here a single relation $R\colon X \longrightarrow\!\!\!\!\!\!+\; Y$ serves as a formal context: its Galois closure yields the underlying complete lattice, while multiplication is induced from the semigroup structure on X.

Remark 4.7. The preceding representation theorem concerns objects. It will later be lifted to a categorical statement. Namely, suitable bonds between residuated relations induce sup-preserving maps between the corresponding concept quantales; imposing compatibility with the descended multiplications singles out precisely the quantale homomorphisms. This is the morphism-level completion of the object-level representation.

5. Multiplicative bonds

Let $R\colon X \longrightarrow\!\!\!\!\!\!+\; Y$ and $S\colon Z \longrightarrow\!\!\!\!\!\!+\; W$ be residuated relations. By Proposition 4.3, the concept lattices

$$\mathsf{M}R \quad \text{and} \quad \mathsf{M}S$$

carry canonical quantale structures. We now give the morphism-level lift of the preceding object-level representation by characterizing those bonds whose induced maps preserve the quantale multiplication.

This is a refinement of Corollary 3.6. There we showed that, for arbitrary formal contexts, bonds are precisely the relational data inducing sup-preserving maps between concept lattices. In the present setting, the relations are residuated and the concept lattices carry quantale structures. Hence the natural question is which bonds correspond, under the correspondence of Corollary 3.6, to homomorphisms of quantales. Proposition 5.3 answers this question by adding exactly one intrinsic compatibility condition, namely multiplicativity of rows.

The mutually inverse order isomorphisms in (3.iii) transport the extent-side multiplication of $\mathsf{M}S$, given by Proposition 4.3, to the intent side. Thus, for $P, Q \in \mathrm{Int}(S)$, write

$$P \otimes_S Q = S^{\uparrow}(S^{\downarrow}(P) * S^{\downarrow}(Q)) = S^{\uparrow}\big(S^{\downarrow}(P) \otimes_S S^{\downarrow}(Q)\big). \tag{5.i}$$

In the second expression, the inner multiplication $\otimes_S$ is the extent-side multiplication of $\mathsf{M}S$. Then

$$S^{\downarrow}(P \otimes_S Q) = S^{\downarrow}(P) \otimes_S S^{\downarrow}(Q), \qquad S^{\uparrow}(C \otimes_S D) = S^{\uparrow}(C) \otimes_S S^{\uparrow}(D)$$

for all $P, Q \in \mathrm{Int}(S)$ and $C, D \in \mathsf{M}S$.

Let $\mathbf{b}\colon R \longrightarrow S$ be a bond. For each $x \in X$, its x-th row

$$\beta_{\mathbf{b}}(x) := \mathbf{b}[x] = \{w \in W \mid x\mathbf{b}w\}$$

is an S-intent (see (B1)). Thus we have a map

$$\beta_{\mathbf{b}}\colon X \longrightarrow \mathrm{Int}(S), \qquad x \mapsto \mathbf{b}[x].$$

Definition 5.1. Let $R\colon X \longrightarrow\!\!\!\!\!\to Y$ and $S\colon Z \longrightarrow\!\!\!\!\!\to W$ be residuated relations. A bond

$$\mathbf{b}\colon R \longrightarrow S$$

is called *multiplicative* if

$$\mathbf{b}[xx'] = \mathbf{b}[x] \otimes_S \mathbf{b}[x']$$

for all $x, x' \in X$.

Equivalently, a bond $\mathbf{b}\colon R \longrightarrow S$ is multiplicative if and only if its row map

$$\beta_{\mathbf{b}}\colon X \longrightarrow \mathrm{Int}(S)$$

is a semigroup homomorphism from X to the semigroup of S-intents equipped with the multiplication $\otimes_S$.

We next prove that this intrinsic row-wise condition is precisely the condition that the induced map between concept lattices preserves the quantale multiplication.

Lemma 5.2. *Let $\mathbf{b}\colon R \longrightarrow S$ be a bond. Then, for every $A \in \mathsf{M}R$,*

$$S^{\uparrow}\mathbf{b}_*(A) = \mathbf{b}^{\uparrow}(A).$$

Moreover, for every $x \in X$,

$$S^{\uparrow}\mathbf{b}_* R^{\downarrow}R^{\uparrow}(\{x\}) = \mathbf{b}[x].$$

Proof. Since $\mathbf{b}$ is a bond, each row $\mathbf{b}[x]$ is an S-intent. Thus, for every $A \subseteq X$,

$$\mathbf{b}^{\uparrow}(A) = \bigcap_{a \in A} \mathbf{b}[a]$$

is an S-intent. Then, it follows from (3.iv) that

$$S^{\uparrow}\mathbf{b}_*(A) = S^{\uparrow}S^{\downarrow}\mathbf{b}^{\uparrow}(A) = \mathbf{b}^{\uparrow}(A).$$

For the second assertion, we use the fact that the columns of $\mathbf{b}$ are R-extents. Note that for every $w \in W$,

$$w \in \mathbf{b}^{\uparrow}R^{\downarrow}R^{\uparrow}(\{x\}) \iff R^{\downarrow}R^{\uparrow}(\{x\}) \subseteq \mathbf{b}^{-1}[w].$$

Since $\mathbf{b}^{-1}[w]$ is an R-extent, this is equivalent to $x \in \mathbf{b}^{-1}[w]$, that is, to $x\mathbf{b}w$. Therefore

$$\mathbf{b}^{\uparrow}R^{\downarrow}R^{\uparrow}(\{x\}) = \mathbf{b}[x].$$

Together with the first assertion, this gives

$$S^{\uparrow}\mathbf{b}_* R^{\downarrow}R^{\uparrow}(\{x\}) = \mathbf{b}[x]. \qquad \square$$

Proposition 5.3. *Let*

$$\mathbf{b}\colon R \longrightarrow S$$

be a bond between residuated relations. Then $\mathbf{b}$ is multiplicative if and only if the induced map

$$\mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}S$$

preserves quantale multiplication; that is,

$$\mathbf{b}_*(A \otimes_R B) = \mathbf{b}_*(A) \otimes_S \mathbf{b}_*(B)$$

for all $A, B \in \mathsf{M}R$.

Proof. **Necessity.** Suppose that $\mathbf{b}$ is multiplicative. Let $x, x' \in X$. Then

$$
\begin{aligned}
& S^{\uparrow}\mathbf{b}_{*}\left(R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_R R^{\downarrow}R^{\uparrow}(\{x'\})\right) \\
&= S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{xx'\}) && \text{(by (2.iii) and Propositions 4.2, 4.3)} \\
&= \mathbf{b}[xx'] && \text{(by Lemma 5.2)} \\
&= \mathbf{b}[x] \otimes_S \mathbf{b}[x'] && \text{(by Definition 5.1)} \\
&= S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\}) && \text{(by Lemma 5.2)} \\
&= S^{\uparrow}\left(\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\})\right). && \text{(by (5.i))}
\end{aligned}
$$

Since $S^{\uparrow}$ is injective on S-extents, we get

$$
\mathbf{b}_{*}\left(R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_R R^{\downarrow}R^{\uparrow}(\{x'\})\right) = \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\}). \tag{5.ii}
$$

Now let $A, B \in \mathsf{M}R$. Since

$$
A = \bigsqcup_{a\in A} R^{\downarrow}R^{\uparrow}(\{a\}), \qquad B = \bigsqcup_{b\in B} R^{\downarrow}R^{\uparrow}(\{b\}),
$$

and since multiplication in both $\mathsf{M}R$ and $\mathsf{M}S$ distributes over arbitrary suprema, we obtain

$$
\begin{aligned}
\mathbf{b}_{*}(A \otimes_R B) &= \mathbf{b}_{*}\Big(\Big(\bigsqcup_{a\in A} R^{\downarrow}R^{\uparrow}(\{a\})\Big) \otimes_R \Big(\bigsqcup_{b\in B} R^{\downarrow}R^{\uparrow}(\{b\})\Big)\Big) \\
&= \mathbf{b}_{*}\Big(\bigsqcup_{a\in A,\ b\in B} \left(R^{\downarrow}R^{\uparrow}(\{a\}) \otimes_R R^{\downarrow}R^{\uparrow}(\{b\})\right)\Big) \\
&= \bigsqcup_{a\in A,\ b\in B} \mathbf{b}_{*}\left(R^{\downarrow}R^{\uparrow}(\{a\}) \otimes_R R^{\downarrow}R^{\uparrow}(\{b\})\right) && \text{(by Proposition 3.4)} \\
&= \bigsqcup_{a\in A,\ b\in B} \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{a\}) \otimes_S \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{b\}) && \text{(by (5.ii))} \\
&= \Big(\bigsqcup_{a\in A} \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{a\})\Big) \otimes_S \Big(\bigsqcup_{b\in B} \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{b\})\Big) \\
&= \mathbf{b}_{*}(A) \otimes_S \mathbf{b}_{*}(B). && \text{(by Proposition 3.4)}
\end{aligned}
$$

Thus $\mathbf{b}_{*}$ preserves quantale multiplication.

Sufficiency. Suppose that $\mathbf{b}_{*}$ preserves quantale multiplication. Let $x, x' \in X$. Then it follows from (2.iii) and Propositions 4.2, 4.3 that

$$
\begin{aligned}
\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{xx'\}) &= \mathbf{b}_{*}\left(R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_R R^{\downarrow}R^{\uparrow}(\{x'\})\right) \\
&= \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{b}[xx'] &= S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{xx'\}) && \text{(by Lemma 5.2)} \\
&= S^{\uparrow}\left(\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S \mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\})\right) \\
&= S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x\}) \otimes_S S^{\uparrow}\mathbf{b}_{*}R^{\downarrow}R^{\uparrow}(\{x'\}) && \text{(by (5.i))} \\
&= \mathbf{b}[x] \otimes_S \mathbf{b}[x']. && \text{(by Lemma 5.2)}
\end{aligned}
$$

Thus $\mathbf{b}$ is multiplicative. □

Remark 5.4. Definition 5.1 formulates multiplicativity intrinsically: the row map

$$
\beta_{\mathbf{b}} \colon X \longrightarrow \mathrm{Int}(S)
$$

is required to be a semigroup homomorphism for the transported intent-side multiplication. Proposition 5.3 shows that this row-wise condition is equivalent to preservation of multiplication by the induced map

$$\mathbf{b}_* \colon \mathsf{M}R \longrightarrow \mathsf{M}S.$$

Since every bond already induces a sup-preserving map, a bond is multiplicative precisely when its induced map is a homomorphism of quantales. Thus multiplicative bonds provide the natural morphism-level lift of the object-level representation.

6. The category of residuated relations and the equivalence theorem

We now define a category

$$\mathbf{ResRel}_{\otimes},$$

whose objects are residuated relations, and whose morphisms are multiplicative bonds. Its identity and composition are those of the underlying bonds in **Bond**. The next two propositions show that these bonds are multiplicative, so this prescription is well-defined.

Proposition 6.1. *Let $R\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ be a residuated relation. Then the relation R itself is a multiplicative bond from R to R. Moreover, the induced map*

$$R_* \colon \mathsf{M}R \longrightarrow \mathsf{M}R$$

is the identity map.

Proof. As observed in the construction of **Bond**, the relation R is the identity bond from R to itself and $R_* = 1_{\mathsf{M}R}$. For $x, x' \in X$, we have

$$\begin{aligned}
R[x] \otimes_R R[x'] &= R^{\uparrow}(R^{\downarrow}(R[x]) * R^{\downarrow}(R[x'])) && \text{(by (5.i))}\\
&= R^{\uparrow}(R^{\downarrow}R^{\uparrow}(\{x\}) * R^{\downarrow}R^{\uparrow}(\{x'\}))\\
&= R^{\uparrow}R^{\downarrow}R^{\uparrow}(R^{\downarrow}R^{\uparrow}(\{x\}) * R^{\downarrow}R^{\uparrow}(\{x'\}))\\
&= R^{\uparrow}R^{\downarrow}R^{\uparrow}(\{x\} * \{x'\}) && \text{(by (2.iii))}\\
&= R^{\uparrow}(\{xx'\})\\
&= R[xx'].
\end{aligned}$$

Thus R is multiplicative. □

Proposition 6.2. *The composite of two multiplicative bonds is again multiplicative.*

Proof. Let

$$\mathbf{b}\colon R \longrightarrow S \qquad \text{and} \qquad \mathbf{c}\colon S \longrightarrow T$$

be multiplicative bonds. By Proposition 5.3, both $\mathbf{b}_*$ and $\mathbf{c}_*$ preserve quantale multiplication. Hence so does $\mathbf{c}_* \circ \mathbf{b}_*$. Since the composition in **Bond** satisfies

$$(\mathbf{c} \circ \mathbf{b})_* = \mathbf{c}_* \circ \mathbf{b}_*,$$

Proposition 5.3 again implies that $\mathbf{c} \circ \mathbf{b}$ is multiplicative. □

Together with the associativity and unit laws in **Bond**, these propositions show that $\mathbf{ResRel}_{\otimes}$ is a category. Forgetting the specified semigroup structure on each domain and retaining the underlying bonds defines a faithful functor

$$\mathbf{ResRel}_{\otimes} \longrightarrow \mathbf{Bond}.$$

It is not in general an inclusion of a subcategory: the same relation may carry different semigroup structures on its domain and thereby define distinct objects of $\mathbf{ResRel}_\otimes$.

Next, we show that the assignment

$$(\mathbf{b}\colon R \longrightarrow S) \mapsto (\mathsf{M}\mathbf{b} = \mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}S)$$

defines a functor

$$\mathsf{M}\colon \mathbf{ResRel}_\otimes \longrightarrow \mathbf{Quant}.$$

Proposition 6.3. $\mathsf{M}\colon \mathbf{ResRel}_\otimes \longrightarrow \mathbf{Quant}$ *is a well-defined functor.*

Proof. By Proposition 4.3, M is well-defined on objects. For a multiplicative bond $\mathbf{b}\colon R \longrightarrow S$, Proposition 3.4 and Proposition 5.3 show that

$$\mathbf{b}_*\colon \mathsf{M}R \longrightarrow \mathsf{M}S$$

is a quantale homomorphism. Thus M is well-defined on morphisms. Moreover, Proposition 6.1 gives

$$\mathsf{M}(1_R) = R_* = 1_{\mathsf{M}R},$$

and the composition law in **Bond** gives

$$\mathsf{M}(\mathbf{c} \circ \mathbf{b}) = (\mathbf{c} \circ \mathbf{b})_* = \mathbf{c}_* \circ \mathbf{b}_* = \mathsf{M}\mathbf{c} \circ \mathsf{M}\mathbf{b}.$$

Therefore M preserves identities and composition. □

Remark 6.4. The functor

$$\mathsf{M}\colon \mathbf{ResRel}_\otimes \longrightarrow \mathbf{Quant}$$

records the passage from residuated relations to their associated concept quantales. It sends a residuated relation to its concept quantale and a multiplicative bond to the quantale homomorphism induced by that bond.

Theorem 6.5. $\mathsf{M}\colon \mathbf{ResRel}_\otimes \longrightarrow \mathbf{Quant}$ *is an equivalence of categories.*

Proof. For all objects R, S of $\mathbf{ResRel}_\otimes$, Corollary 3.6 and Proposition 5.3 yield a bijection

$$\mathbf{ResRel}_\otimes(R, S) \longrightarrow \mathbf{Quant}(\mathsf{M}R, \mathsf{M}S), \qquad \mathbf{b} \mapsto \mathbf{b}_*.$$

Thus M is full and faithful. Theorem 4.4 shows that every quantale is isomorphic to $\mathsf{M}R$ for some object R of $\mathbf{ResRel}_\otimes$, so M is essentially surjective. Hence M is an equivalence of categories. □

Remark 6.6. The equivalence above is the natural morphism-level completion of the object-level representation theorem. Theorem 4.4 says that every quantale can be represented as the concept lattice of a residuated relation, and characterizes such representations by sup-dense and inf-dense maps. Theorem 6.5 identifies the homomorphisms between the represented quantales precisely with the maps induced by multiplicative bonds. Thus residuated relations and multiplicative bonds provide a categorical presentation of quantales.

7. The semigroup-object interpretation

The category **Bond** carries more structure than its equivalence with **Sup** alone records. Mori established a lattice tensor on **Bond**, denoted here by $\boxtimes$, and a strong monoidal equivalence

$$(\mathbf{Bond}, \boxtimes) \simeq_\otimes (\mathbf{Sup}, \boxtimes),$$

where the tensor on the right is the standard tensor product of complete sup-lattices [26]; see also [41, Theorem 27]. With the non-unital convention adopted in this paper, a quantale is a *semigroup object* [24] in

$(\mathbf{Sup}, \boxtimes)$, while a unital quantale is a monoid object. For a monoidal category $(\mathcal{C}, \boxtimes)$, we write $\mathbf{SGrp}(\mathcal{C}, \boxtimes)$ for its category of semigroup objects.

Via the canonical isomorphism of categories between **Bond** and $\mathbf{B}(\mathbf{Rel})$ (see Remark 3.7), we transport Mori's monoidal structure from **Bond** to $\mathbf{B}(\mathbf{Rel})$. Equipped with the transported tensor $\boxtimes$, this isomorphism is strong monoidal by construction.

Proposition 7.1. *There are equivalences of categories*

$$\mathbf{ResRel}_{\otimes} \simeq \mathbf{Quant} \simeq \mathbf{SGrp}(\mathbf{Bond}, \boxtimes) \simeq \mathbf{SGrp}(\mathbf{B}(\mathbf{Rel}), \boxtimes).$$

Proof. Theorem 6.5 gives the first equivalence. The strong monoidal equivalence above induces an equivalence between the corresponding categories of semigroup objects. The semigroup objects of $(\mathbf{Sup}, \boxtimes)$ are precisely the non-unital quantales. Hence

$$\mathbf{SGrp}(\mathbf{Bond}, \boxtimes) \simeq \mathbf{SGrp}(\mathbf{Sup}, \boxtimes) = \mathbf{Quant}.$$

Finally, the transported strong monoidal isomorphism

$$(\mathbf{Bond}, \boxtimes) \cong_{\otimes} (\mathbf{B}(\mathbf{Rel}), \boxtimes)$$

induces an equivalence

$$\mathbf{SGrp}(\mathbf{Bond}, \boxtimes) \simeq \mathbf{SGrp}(\mathbf{B}(\mathbf{Rel}), \boxtimes),$$

which gives the final equivalence. ☐

This proposition is a categorical interpretation of the main theorem, rather than a replacement for its relation-level proof. An object of $\mathbf{ResRel}_{\otimes}$ is a concrete residuated presentation

$$R \colon X \twoheadrightarrow Y$$

whose object set X carries a semigroup structure; its attribute set Y is only a set. The residual equations use the multiplication on X to make the Galois closure

$$\mathsf{j}_R = R^{\downarrow} R^{\uparrow}$$

into a quantic nucleus, and hence to make the concept lattice $\mathsf{M}R$ into a quantale. The row condition on a multiplicative bond then says exactly that the induced sup-lattice map preserves this descended multiplication.

By contrast, a semigroup object of $(\mathbf{Bond}, \boxtimes)$ is an internal multiplication in the bond category. The proposition says that, up to the displayed categorical equivalences, these internal semigroup objects are represented by the concrete residuated presentations described in this paper. It does not say that an arbitrary internal semigroup object reconstructs, on its original object and attribute sets, a semigroup operation and residual maps of the displayed form. This distinction is the value of the relation-level description: the residual equations and the row criterion supply explicit object–attribute data whose Galois quotient realizes the abstract internal multiplication.

8. Pseudogroup completions and étale groupoids

Let S be an *abstract complete pseudogroup* (see [32, Definitions 2.8 and 2.9]), namely a complete and infinitely distributive *inverse semigroup* [7, 28]. For $s \in S$, write s^{-1} for its unique inverse. The natural order on S is given by

$$s \leqslant t \iff s = et \quad \text{for some idempotent } e \in S.$$

A subset of S is *compatible* if $s^{-1}t$ and st^{-1} are idempotent for all s, t in the subset. Completeness means that every compatible subset, including the empty subset, has a supremum; in particular, S is an inverse monoid. Infinite distributivity means that multiplication distributes over these suprema. The construction below realizes the enveloping quantale of S as a concept quantale, thereby giving an FCA realization of the pseudogroup completion (see [32, Definition 3.22 and Lemma 3.23]). Let $\mathcal{L}^\vee(S)$ be the set of compatibly closed ideals of S, namely the subsets that are downward closed in the natural order and closed under compatible suprema. For $A \subseteq S$, put

$$\mathrm{cl}_S(A) = \bigcap\{U \in \mathcal{L}^\vee(S) \mid A \subseteq U\}.$$

For $U, V \in \mathcal{L}^\vee(S)$, write

$$U \star V = \mathrm{cl}_S(U * V), \qquad U^\dagger = \{u^{-1} \mid u \in U\}.$$

Thus the dagger on $\mathcal{L}^\vee(S)$ denotes the pointwise extension of the inverse operation on S.

Recall that an *involution* on a quantale Q is a sup-preserving map $a \mapsto a^\dagger$ such that $a^{\dagger\dagger} = a$ and $(a*b)^\dagger = b^\dagger * a^\dagger$ for all $a, b \in Q$. A *support* on a unital involutive quantale Q, with unit e, is a sup-preserving map $\varsigma\colon Q \longrightarrow Q$ satisfying

$$\varsigma(a) \leqslant e, \qquad \varsigma(a) \leqslant a * a^\dagger, \qquad a \leqslant \varsigma(a) * a.$$

A *frame* is a complete lattice in which finite meets distribute over arbitrary suprema. A *supported quantal frame* is a unital involutive quantale whose underlying complete lattice is a frame and which is equipped with a support. An *inverse quantal frame* is a supported quantal frame whose top element is the supremum of its partial units, where u is a partial unit when $u * u^\dagger \leqslant e$ and $u^\dagger * u \leqslant e$ (see [32, Definition 4.10]).

Then $\mathcal{L}^\vee(S)$ is a unital involutive quantal frame, with unit $\downarrow 1_S$ and multiplication $\star$ (see [32, Lemma 3.23]).

Definition 8.1. With S as above, the *pseudogroup membership context* of S is

$$R_S\colon S \relbar\joinrel\mapsto \mathcal{L}^\vee(S), \qquad sR_SU \iff s \in U,$$

with object semigroup S and attribute set $\mathcal{L}^\vee(S)$.

This is the residual-stable membership construction of Example A.6 with $\mathcal{T} = \mathcal{L}^\vee(S)$; the next proposition verifies the required residual stability in this case.

Proposition 8.2. *The relation R_S is residuated. Its residual operations are*

$$U \swarrow t = \{s \in S \mid st \in U\}, \qquad s \searrow U = \{t \in S \mid st \in U\}.$$

Moreover,

$$sR_SU \iff s^{-1}R_SU^\dagger.$$

Proof. For $U \in \mathcal{L}^\vee(S)$ and $t \in S$, the set $U \swarrow t$ is downward closed. If $E \subseteq U \swarrow t$ is compatible, then Et is compatible and

$$(\bigvee E)t = \bigvee(Et) \in U.$$

Thus $U \swarrow t$ is compatibly closed. Similarly, if $E \subseteq s \searrow U$ is compatible, then sE is compatible and

$$s(\bigvee E) = \bigvee(sE) \in U,$$

so $s \searrow U \in \mathcal{L}^\vee(S)$. The defining equivalences are

$$(st)R_SU \iff st \in U \iff s \in U \swarrow t \iff sR_S(U \swarrow t)$$

and, symmetrically, $(st)R_SU \iff tR_S(s \searrow U)$. Finally, $U^\dagger \in \mathcal{L}^\vee(S)$ and $s \in U$ if and only if $s^{-1} \in U^\dagger$. $\square$

Proposition 8.3. *For every $A \subseteq S$,*

$$R_S^\downarrow R_S^\uparrow(A) = \mathrm{cl}_S(A).$$

Consequently, the extents of R_S are precisely the compatibly closed ideals, and

$$\mathsf{M}R_S \cong \mathcal{L}^\vee(S)$$

as unital involutive quantales.

Proof. One has
$$R_S^{\uparrow}(A) = \{U \in \mathcal{L}^{\vee}(S) \mid A \subseteq U\};$$
hence its lower transform is exactly $\mathrm{cl}_S(A)$. Thus the extents are the members of $\mathcal{L}^{\vee}(S)$. For closed ideals U, V,
$$U \otimes_{R_S} V = R_S^{\downarrow} R_S^{\uparrow}(U * V) = \mathrm{cl}_S(U * V) = U \star V.$$
The unit $\downarrow 1_S$ and the involution $U \mapsto U^{\dagger}$ are therefore those of $\mathcal{L}^{\vee}(S)$, which proves the assertion. □

Corollary 8.4. *For every abstract complete pseudogroup S, the concept quantale $\mathsf{M}R_S$ is an inverse quantal frame. Hence there is a localic étale groupoid G_S for which*
$$\mathsf{M}R_S \cong \mathcal{O}(G_S)$$
as unital involutive quantales, where $\mathcal{O}(G_S)$ denotes the quantale of opens of the arrow locale of G_S.

Proof. Proposition 8.3 identifies $\mathsf{M}R_S$ with $\mathcal{L}^{\vee}(S)$. The latter is an inverse quantal frame (see [32, Definition 3.16, Lemma 3.23 and Definition 4.10]), and the groupoid statement follows from the inverse-quantal-frame reconstruction and recovery results (see [32, Theorems 4.19 and 5.11]). □

Here a homomorphism of abstract complete pseudogroups means a monoid homomorphism that preserves suprema of compatible subsets. Let $f \colon S \longrightarrow T$ be such a homomorphism. Its universal extension
$$\mathcal{L}^{\vee}(f) \colon \mathcal{L}^{\vee}(S) \longrightarrow \mathcal{L}^{\vee}(T), \qquad U \mapsto \mathrm{cl}_T(f[U])$$
is a unital involutive quantale homomorphism (see [32, Theorem 3.25 and Corollary 3.26]). The associated multiplicative bond
$$\mathbf{b}_f \colon R_S \longrightarrow R_T$$
has the particularly transparent form
$$s\, \mathbf{b}_f\, V \iff f(s) \in V, \qquad (s \in S,\ V \in \mathcal{L}^{\vee}(T)).$$
Indeed,
$$R_S^{\downarrow} R_S^{\uparrow}(\{s\}) = \downarrow s \qquad \text{and} \qquad \mathcal{L}^{\vee}(f)(\downarrow s) = \downarrow f(s).$$
Since V is downward closed, the bond obtained from $\mathcal{L}^{\vee}(f)$ by Proposition 3.5 has precisely the displayed incidence relation. It is multiplicative by Proposition 5.3.

A. Examples of residuated relations

The examples below illustrate four recurrent sources of residuated relations: order comparisons, membership in residual-stable families of subsets, quantale-valued observations, and equations determined by multiplicative invariants. In each case we display the residual maps and identify the resulting concept quantale. Some examples are special cases of more general ones, but are retained because the residual operations have familiar interpretations in their respective settings.

A.1. Order-induced relations

Example A.1 (Dense ordered presentations)**.** Let Q be a quantale, and let $A \subseteq Q$ be a subsemigroup that is both sup-dense and inf-dense in Q. Suppose that A, with the order inherited from Q, is a residuated ordered semigroup. Then the restricted order relation
$$\leqslant \,:\, A \longrightarrow\!\!\!\!\!\!\mapsto A$$
is residuated. The inclusion $A \longrightarrow Q$ is at once a sup-dense semigroup homomorphism and an inf-dense map, so Theorem 4.4 gives
$$\mathsf{M}(A, A, \leqslant) \cong Q$$

as quantales.

For example, take $Q = ([0,1], \wedge)$ and $A = \mathbb{Q} \cap [0,1]$. The residual in A is

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{if } b < a. \end{cases}$$

Thus the rational order context presents the quantale $([0,1], \wedge)$.

As a non-idempotent example, consider the Lawvere quantale $Q = ([0,\infty], \geqslant, +)$ [22], ordered oppositely to the usual order, and put $A = \mathbb{Q}_{\geqslant 0} \cup \{\infty\}$. This is a sup-dense and inf-dense subsemigroup. Its residual is truncated subtraction:

$$b \ominus a = \begin{cases} 0 & \text{if } b \leqslant a, \\ b - a & \text{if } a < b < \infty, \\ \infty & \text{if } b = \infty \text{ and } a < \infty, \end{cases}$$

where the displayed inequalities use the usual order. Hence $(A, A, \geqslant)$ presents $([0,\infty], \geqslant, +)$.

Example A.2 (A frame from a meet-semilattice basis)**.** Let L be a frame and let $B \subseteq L$ be a meet-semilattice basis [18]. Regard B as a semigroup under finite meet, and define

$$R\colon B \looparrowright L, \qquad bRu \iff b \leqslant u.$$

If $\to$ denotes the Heyting implication of L, put

$$u \swarrow b' = b' \to u, \qquad b \searrow u = b \to u.$$

Then

$$b \wedge b' \leqslant u \iff b \leqslant b' \to u \iff b' \leqslant b \to u,$$

so R is residuated. The inclusion $B \longrightarrow L$ is sup-dense, while $1_L\colon L \longrightarrow L$ is inf-dense. Therefore

$$\mathsf{M}R \cong L$$

as quantales. Thus a frame can be presented by its basic elements as objects and all its elements as attributes.

Example A.3 (Quantic nuclei as restricted attributes)**.** Let Q be a quantale, let j be a quantic nucleus on Q, and let $f\colon X \longrightarrow Q$ be a sup-dense semigroup homomorphism. Take the fixed-point set Q_{j} as the attribute set and define

$$R_{\mathsf{j}}\colon X \looparrowright Q_{\mathsf{j}}, \qquad xR_{\mathsf{j}}y \iff f(x) \leqslant y.$$

For $y \in Q_{\mathsf{j}}$ and $x, x' \in X$, put

$$y \swarrow x' = y \swarrow f(x'), \qquad x \searrow y = f(x) \searrow y,$$

where the operations on the right are the residuals in Q. These elements belong to Q_{j}. Indeed, if $c = y \swarrow f(x')$, then

$$\mathsf{j}(c) * f(x') \leqslant \mathsf{j}(c) * \mathsf{j}f(x') \leqslant \mathsf{j}(c * f(x')) \leqslant \mathsf{j}(y) = y,$$

and hence $\mathsf{j}(c) \leqslant c$; the reverse inequality is automatic. The other residual is treated similarly. It follows that R_{j} is residuated.

For every $A \subseteq X$, its Galois closure is

$$R_{\mathsf{j}}^{\downarrow} R_{\mathsf{j}}^{\uparrow}(A) = \left\{ x \in X \;\middle|\; f(x) \leqslant \mathsf{j}\Big(\bigvee f[A]\Big) \right\}.$$

The map $x \mapsto \mathsf{j}f(x)$ is a sup-dense semigroup homomorphism $X \longrightarrow Q_{\mathsf{j}}$, and the identity map of Q_{j} is inf-dense. Hence

$$\mathsf{M}R_{\mathsf{j}} \cong Q_{\mathsf{j}}$$

as quantales. This realizes a quantale quotient by retaining the original objects and restricting the attributes to the closed elements of the nucleus.

A.2. Membership relations

Example A.4 (The powerset membership context)**.** Let X be a semigroup. The membership relation

$$R_X \colon X \twoheadrightarrow \mathsf{P}X, \qquad xR_XA \iff x \in A$$

is residuated, with

$$A \swarrow x' = \{x \in X \mid xx' \in A\}, \qquad x \searrow A = \{x' \in X \mid xx' \in A\}.$$

Indeed, $(xx')R_XA$ holds precisely when $x \in A \swarrow x'$, equivalently when $x' \in x \searrow A$. Its Galois closure is the identity on $\mathsf{P}X$, since every subset is itself an attribute. Hence

$$\mathsf{M}R_X \cong \mathsf{P}X$$

as quantales. This is the unclosed limiting case of the membership construction.

Example A.5 (Quotient attributes from a semigroup homomorphism)**.** Let

$$h \colon X \longrightarrow M$$

be a surjective semigroup homomorphism. Put

$$\mathcal{T}_h = \{h^{-1}(B) \mid B \subseteq M\}.$$

For $B \subseteq M$ and $x, x' \in X$, the point residuals are

$$h^{-1}(B) \swarrow x' = h^{-1}(\{m \in M \mid mh(x') \in B\})$$

and

$$x \searrow h^{-1}(B) = h^{-1}(\{m \in M \mid h(x)m \in B\}).$$

Thus the displayed residuals belong to $\mathcal{T}_h$, and the membership relation

$$R_h \colon X \twoheadrightarrow \mathcal{T}_h, \qquad xR_hh^{-1}(B) \iff h(x) \in B$$

is residuated. Its Galois closure is

$$R_h^{\downarrow}R_h^{\uparrow}(A) = h^{-1}(h[A])$$

for every $A \subseteq X$. The extents are therefore exactly the saturated subsets $h^{-1}(B)$, and

$$\mathsf{M}R_h \cong \mathsf{P}M$$

as quantales. Hence a surjective semigroup homomorphism produces a formal-concept presentation of the powerset quantale of its quotient semigroup.

Example A.6 (Residual-stable families of attributes)**.** Let X be a semigroup and let $\mathcal{T}$ be a family of subsets of X that is stable under the point residuals

$$T \swarrow x = \{u \in X \mid ux \in T\} \in \mathcal{T}, \qquad x \searrow T = \{u \in X \mid xu \in T\} \in \mathcal{T}$$

for all $T \in \mathcal{T}$ and $x \in X$. Then the membership relation

$$R_{\mathcal{T}} \colon X \twoheadrightarrow \mathcal{T}, \qquad xR_{\mathcal{T}}T \iff x \in T$$

is residuated. Its Galois closure

$$\mathsf{j}_{\mathcal{T}} := R_{\mathcal{T}}^{\downarrow}R_{\mathcal{T}}^{\uparrow}$$

is

$$\mathsf{j}_{\mathcal{T}}(A) = \bigcap\{T \in \mathcal{T} \mid A \subseteq T\},$$

where the empty intersection is X. Proposition 4.2 therefore says that $\mathsf{j}_{\mathcal{T}}$ is a quantic nucleus, and its fixed points form a quantale.

This construction allows one to choose a proper family of attributes rather than all subsets of X. Residual stability ensures that the resulting closure respects multiplication. The pseudogroup membership context of Section 8 is an instance of this mechanism.

Remark A.7 (Formal languages and trace specifications)**.** For the free monoid $X = \Sigma^*$, the residuals in the powerset membership context are the familiar right and left language quotients [11]:

$$L \swarrow v = L/v = \{u \in \Sigma^* \mid uv \in L\}, \qquad u \searrow L = u\backslash L = \{v \in \Sigma^* \mid uv \in L\}.$$

Choosing all languages gives the language quantale $\mathsf{P}(\Sigma^*)$. Fixing a surjective monoid homomorphism $h\colon \Sigma^* \longrightarrow M$ instead and using only inverse images of subsets of M gives Example A.5, with closure $A \mapsto h^{-1}(h[A])$ and concept quantale $\mathsf{P}M$. By contrast, when Σ is finite, allowing all regular languages still gives the identity closure: for $w \notin A$, the regular language $\Sigma^* \setminus \{w\}$ contains A but not w. The same quotient formulas apply to a semigroup of program traces, as in trace semantics [16], where L/v describes the prefixes that satisfy L after the continuation v.

Example A.8 (Ideal and radical-ideal attributes)**.** Let A be a commutative ring, regarded as a multiplicative semigroup. First consider the membership relation

$$R_{\mathrm{Idl}}\colon A \longrightarrow\!\!\!\!\!\!\mapsto \mathrm{Idl}(A), \qquad aR_{\mathrm{Idl}}I \iff a \in I,$$

where the attribute set is the set of ideals. For $I \in \mathrm{Idl}(A)$ and $b \in A$, both residual attributes are the *colon ideal of I by b* [3],

$$I \swarrow b = (I : b) = \{a \in A \mid ab \in I\}, \qquad b \searrow I = (I : b).$$

Thus R_{Idl} is residuated. Its induced closure sends $S \subseteq A$ to the ideal generated by S:

$$R^{\downarrow}_{\mathrm{Idl}}R^{\uparrow}_{\mathrm{Idl}}(S) = (S).$$

Consequently,

$$\mathsf{M}R_{\mathrm{Idl}} \cong \mathrm{Idl}(A)$$

as quantales, with multiplication given by ideal product.

Now restrict the attributes to radical ideals. If I is radical, then $(I : b)$ is radical. Indeed, if $c^n \in (I : b)$, then $c^n b \in I$, and

$$(cb)^n = (c^n b)b^{n-1} \in I.$$

It follows that $cb \in I$, and hence $c \in (I : b)$. Therefore the same colon ideals give residual attributes for

$$R_{\mathrm{rad}}\colon A \longrightarrow\!\!\!\!\!\!\mapsto \mathrm{RIdl}(A), \qquad aR_{\mathrm{rad}}I \iff a \in I.$$

Its closure is

$$R^{\downarrow}_{\mathrm{rad}}R^{\uparrow}_{\mathrm{rad}}(S) = \sqrt{(S)},$$

and hence

$$\mathsf{M}R_{\mathrm{rad}} \cong \mathrm{RIdl}(A)$$

as quantales, with multiplication $I \otimes J = \sqrt{IJ}$.

Remark A.9 (Affine-variety form)**.** Let $A = K[x_1, \ldots, x_n]$, where K is algebraically closed, and take the affine algebraic subsets of K^n as attributes. Define

$$fR_{\mathrm{Zar}}V \iff f|_V = 0.$$

For $g \in A$, put

$$V \swarrow g = V(I(V) : g), \qquad g \searrow V = V(I(V) : g).$$

The ideal $(I(V) : g)$ is radical by the preceding example. The Hilbert Nullstellensatz [15] therefore gives

$$fR_{\mathrm{Zar}}(V \swarrow g) \iff f \in (I(V) : g) \iff fg \in I(V) \iff (fg)R_{\mathrm{Zar}}V,$$

and similarly for the other residual. The induced closure is $S \mapsto \sqrt{(S)}$. Thus this relation presents the radical-ideal quantale, or equivalently the lattice of affine algebraic sets in the opposite order, with multiplication corresponding to union of algebraic sets.

Example A.10 (Closed-set attributes in a topological semigroup)**.** Let X be a *topological semigroup* [6], and let $\mathsf{C}X$ be the set of closed subsets of X. The membership relation

$$R_{\mathrm{cl}}\colon X \longrightarrow\!\!\!\!\!\rightarrow \mathsf{C}X, \qquad xR_{\mathrm{cl}}F \iff x \in F$$

is residuated. Indeed, for $F \in \mathsf{C}X$ and $a, b \in X$, the residual attributes

$$F \swarrow b = \{a \in X \mid ab \in F\}, \qquad a \searrow F = \{b \in X \mid ab \in F\}$$

are closed, since they are inverse images of F under continuous right and left translations. Moreover,

$$R_{\mathrm{cl}}^{\downarrow}R_{\mathrm{cl}}^{\uparrow}(A) = \overline{A}$$

for all $A \subseteq X$. Consequently,

$$\mathsf{M}R_{\mathrm{cl}} \cong \mathsf{C}X$$

as quantales, with multiplication

$$F \otimes G = \overline{F * G}.$$

Thus ordinary topological closure is obtained directly as the Galois closure of a natural residuated membership relation.

Example A.11 (Closed linear subspaces of operator algebras)**.** Let A be a C^*-algebra, regarded as a multiplicative semigroup, and let $\operatorname{Max} A$ be the set of norm-closed complex linear subspaces of A [20]. Define

$$R_{\mathrm{Max}}\colon A \longrightarrow\!\!\!\!\!\rightarrow \operatorname{Max} A, \qquad aR_{\mathrm{Max}}M \iff a \in M.$$

For $N \in \operatorname{Max} A$ and $a, b \in A$, put

$$N \swarrow b = \{a \in A \mid ab \in N\}, \qquad a \searrow N = \{b \in A \mid ab \in N\}.$$

These are norm-closed complex linear subspaces, being inverse images of N under bounded linear maps. Hence R_{Max} is residuated, and its Galois closure is

$$R_{\mathrm{Max}}^{\downarrow}R_{\mathrm{Max}}^{\uparrow}(S) = \overline{\operatorname{span}(S)}.$$

It follows that

$$\mathsf{M}R_{\mathrm{Max}} \cong \operatorname{Max} A$$

as quantales, with multiplication

$$M \otimes N = \overline{\operatorname{span}(\{mn \mid m \in M,\ n \in N\})}.$$

The same argument applies to a von Neumann algebra with weak-* closed linear subspaces as attributes [10]; see [29] for the weak-operator version. Since multiplication by a fixed element is weak-* continuous, the residual attributes are weak-* closed, and the induced closure is weak-* closed linear span. If A is a commutative C^*-algebra, one may instead use norm-closed ideals as attributes; the resulting concept quantale is the quantale of closed ideals, with multiplication $I \otimes J = \overline{IJ}$. For a general noncommutative C^*-algebra, closed two-sided ideals need not be stable under both point residuals.

A.3. Observation-induced relations

Example A.12 (Quantale-valued observations)**.** Let X be a semigroup, let Q be a quantale, and let

$$\Phi\colon X \longrightarrow Q$$

be a semigroup homomorphism. For the interpretation of quantales as algebras of observations, see [30]. Define

$$R_{\Phi}\colon X \longrightarrow\!\!\!\!\!\rightarrow Q, \qquad xR_{\Phi}q \iff \Phi(x) \leqslant q.$$

The quantale residuals give

$$q \swarrow x' = q \swarrow \Phi(x'), \qquad x \searrow q = \Phi(x) \searrow q,$$

and hence R_Φ is residuated. For every $A \subseteq X$,

$$R_\Phi^{\downarrow} R_\Phi^{\uparrow}(A) = \left\{ x \in X \;\middle|\; \Phi(x) \leqslant \bigvee_{a \in A} \Phi(a) \right\}.$$

Consequently, $\mathsf{M}R_\Phi$ is isomorphic to the subquantale

$$Q_\Phi = \left\{ \bigvee_{a \in A} \Phi(a) \;\middle|\; A \subseteq X \right\}$$

of Q. Thus every multiplicative representation of a semigroup in a quantale canonically yields a residuated relation.

Remark A.13 (Relational semantics and group actions)**.** Let S be a set and take $Q = \mathsf{P}(S \times S)$ with relational composition in the convention

$$P \circ Q = \{(s,t) \mid \text{there exists } u \in S \text{ with } (s,u) \in P \text{ and } (u,t) \in Q\}.$$

This is the standard quantale of binary relations; see [12] for the relational background. A semigroup homomorphism

$$\Phi \colon X \longrightarrow \mathsf{P}(S \times S)$$

may be viewed as a compositional relational semantics for the elements of X [1]. The preceding relation becomes

$$x R_\Phi T \iff \Phi(x) \subseteq T,$$

and its concept quantale is the subquantale generated by the relational behaviours $\Phi(x)$. In particular, a right action of a group G on S gives such a homomorphism by sending g to the graph of $s \mapsto s \cdot g$.

Remark A.14 (Closed set-valued observations)**.** Let X be a semigroup, let Y be a topological semigroup, and let $\Phi \colon X \longrightarrow \mathsf{P}Y$ be a semigroup homomorphism. For general background on set-valued maps, see [4]. Let $\mathsf{C}Y$ be the set of closed subsets of Y, and define

$$R_\Phi^{\mathrm{cl}} \colon X \longrightarrow\!\!\!\!\!\!\!\;\mapsto \mathsf{C}Y, \qquad x R_\Phi^{\mathrm{cl}} C \iff \Phi(x) \subseteq C$$

for $x \in X$ and $C \in \mathsf{C}Y$. For $C \in \mathsf{C}Y$ and $x, x' \in X$, put

$$C \swarrow x' = \{y \in Y \mid y\Phi(x') \subseteq C\}, \qquad x \searrow C = \{y \in Y \mid \Phi(x)y \subseteq C\}.$$

These two residual attributes are closed, being intersections of inverse images of C under continuous right and left translations, respectively. Since $\Phi(xx') = \Phi(x) * \Phi(x')$, they satisfy

$$(xx') R_\Phi^{\mathrm{cl}} C \iff x R_\Phi^{\mathrm{cl}} (C \swarrow x') \iff x' R_\Phi^{\mathrm{cl}} (x \searrow C).$$

Thus R_Φ^{cl} is residuated. Writing $\Phi[A] = \bigcup_{a \in A} \Phi(a)$, its Galois closure is

$$\mathsf{j}_{R_\Phi^{\mathrm{cl}}}(A) = \{x \in X \mid \Phi(x) \subseteq \overline{\Phi[A]}\}.$$

Applying $\mathsf{j}_{R_\Phi^{\mathrm{cl}}}$ to A does not change $\overline{\Phi[A]}$. Hence the map

$$\mathsf{Fix}\big(\mathsf{j}_{R_\Phi^{\mathrm{cl}}}\big) \longrightarrow \mathsf{C}Y, \qquad A \longmapsto \overline{\Phi[A]},$$

identifies the fixed-point quantale with the subquantale

$$\{\overline{\Phi[A]} \mid A \subseteq X\} \subseteq \mathsf{C}Y,$$

whose suprema are closures of unions and whose multiplication is $F \otimes G = \overline{F * G}$.

A.4. Multiplicative invariants

Example A.15 (Group-valued multiplicative invariants)**.** Let X be a semigroup, let G be a group, let $h\colon X \longrightarrow G$ be a semigroup homomorphism, and fix $q \in G$. Define

$$R_q\colon X \longrightarrow\!\!\!\!\!\!\rightarrow G, \qquad xR_q g \iff h(x)g = q.$$

For $g \in G$ and $x, x' \in X$, put

$$g \swarrow x' = h(x')g, \qquad x \searrow g = gq^{-1}h(x)q.$$

A direct calculation gives

$$(xx')R_q g \iff xR_q(g \swarrow x') \iff x'R_q(x \searrow g),$$

so R_q is residuated. If G is commutative, the second formula simplifies to $x \searrow g = gh(x)$.

For $a \in h[X]$, let $F_a = h^{-1}(a)$. Then

$$R_q^{\uparrow}(F_a) = \{a^{-1}q\}, \qquad R_q^{\downarrow}R_q^{\uparrow}(F_a) = F_a,$$

so every nonempty fibre of h is an extent. A subset meeting two distinct fibres has no common attribute and therefore closes to X. Moreover, for $a, b \in h[X]$,

$$F_a \otimes_{R_q} F_b = F_{ab}.$$

Thus the concept quantale retains the multiplication of the invariant values while identifying elements in the same fibre.

Example A.16 (Determinant tests and singular collapse)**.** Let K be a field, let $n \geqslant 1$, regard $M_n(K)$ as a multiplicative semigroup, and take K as the attribute set. Fix $q \in K^* = K \setminus \{0\}$, and define

$$\varphi_q\colon M_n(K) \longrightarrow\!\!\!\!\!\!\rightarrow K, \qquad A\varphi_q\lambda \iff \det(A)\lambda = q.$$

On $GL_n(K)$, this is the preceding construction for the group-valued invariant $\det\colon GL_n(K) \longrightarrow K^*$. The present context also includes singular matrices and the zero scalar.

For $A, A' \in M_n(K)$ and $\lambda \in K$, put

$$\lambda \swarrow A' = \det(A')\lambda, \qquad A \searrow \lambda = \lambda \det(A).$$

Since K is commutative and the determinant is multiplicative [2],

$$(AA')\varphi_q\lambda \iff A\varphi_q(\lambda \swarrow A') \iff A'\varphi_q(A \searrow \lambda).$$

Hence φ_q is a residuated relation.

For $\mu \in K$, write

$$D_\mu = \{A \in M_n(K) \mid \det(A) = \mu\}.$$

If $\mu \in K^*$, then

$$\varphi_q^{\uparrow}(D_\mu) = \{q\mu^{-1}\} \qquad \text{and} \qquad \varphi_q^{\downarrow}\varphi_q^{\uparrow}(D_\mu) = D_\mu.$$

On the other hand,

$$\varphi_q^{\uparrow}(D_0) = \varnothing \qquad \text{and} \qquad \varphi_q^{\downarrow}\varphi_q^{\uparrow}(D_0) = M_n(K),$$

because no scalar satisfies $0\lambda = q$. More generally, for every $S \subseteq M_n(K)$,

$$\varphi_q^{\downarrow}\varphi_q^{\uparrow}(S) = \begin{cases} \varnothing & \text{if } S = \varnothing, \\ D_\mu & \text{if } S \neq \varnothing \text{ and } \det(A) = \mu \in K^* \text{ for all } A \in S, \\ M_n(K) & \text{otherwise.} \end{cases}$$

Indeed, a nonempty family of matrices has a common scalar attribute precisely when all its members have the same nonzero determinant. Consequently, the extents of φ_q are exactly

$$\varnothing, \qquad D_\mu \quad (\mu \in K^*), \qquad M_n(K).$$

For $\mu, \nu \in K^*$, one has

$$D_\mu * D_\nu = D_{\mu\nu}.$$

The inclusion $D_\mu * D_\nu \subseteq D_{\mu\nu}$ follows from the multiplicativity of the determinant. Conversely, for $C \in D_{\mu\nu}$, choose $A \in D_\mu$ and put $B = A^{-1}C$; then $B \in D_\nu$ and $C = AB$. Therefore

$$D_\mu \otimes_{\varphi_q} D_\nu = D_{\mu\nu}.$$

Moreover, $\varnothing$ is multiplicatively absorbing, while

$$M_n(K) \otimes_{\varphi_q} E = E \otimes_{\varphi_q} M_n(K) = M_n(K)$$

for every nonempty extent E. Thus the nonzero determinant fibres reproduce the group multiplication of K^*, whereas every singular matrix is indistinguishable from the top extent under the chosen nonzero scalar attributes.

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the authors used Codex in ChatGPT Work (OpenAI) to assist with manuscript organization, language editing, consistency checking, and submission-readiness review. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

References

[1] S. Abramsky and S. Vickers. Quantales, observational logic and process semantics. *Mathematical Structures in Computer Science*, 3(2):161–227, 1993.

[2] M. Artin. *Algebra*. Pearson, Hoboken, second edition, 2010.

[3] M. F. Atiyah and I. G. MacDonald. *Introduction To Commutative Algebra*. Addison–Wesley, Reading, 1969.

[4] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, Boston, 1990.

[5] C. Brown and D. Gurr. A representation theorem for quantales. *Journal of Pure and Applied Algebra*, 85(1):27–42, 1993.

[6] J. H. Carruth, J. A. Hildebrant, and R. J. Koch. *The Theory of Topological Semigroups*. Marcel Dekker, New York, 1983.

[7] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups, Part 1*, volume 7 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, 1961.

[8] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, second edition, 2002.

[9] B. J. Day and S. Lack. Limits of small functors. *Journal of Pure and Applied Algebra*, 210(3):651–663, 2007.

[10] J. M. Egger and D. Kruml. Girard couples of quantales. *Applied Categorical Structures*, 18(2):123–133, 2010.

[11] S. Eilenberg. *Automata, Languages, and Machines*, volume 59, Part A of *Pure and Applied Mathematics*. Academic Press, New York, 1974.

[12] P. J. Freyd and A. Scedrov. *Categories, Allegories*, volume 39 of *North-Holland Mathematical Library*. Elsevier Science, Amsterdam, 1990.

[13] B. Ganter. Relational Galois connections. In S. O. Kuznetsov and S. Schmidt, editors, *Formal Concept Analysis*, volume 4390 of *Lecture Notes in Computer Science*, pages 1–17. Springer, Berlin–Heidelberg, 2007.

[14] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer, Berlin–Heidelberg, 1999.

[15] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, New York, 1977.

[16] C. A. R. Hoare. *Communicating Sequential Processes*. Prentice Hall, Hoboken, 1985.

[17] J. M. Howie. *Fundamentals of Semigroup Theory*, volume 12 of *London Mathematical Society Monographs New Series*. Clarendon Press, Oxford, 1995.

[18] P. T. Johnstone. *Stone Spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[19] G. M. Kelly and V. Schmitt. Notes on enriched categories with colimits of some class. *Theory and Applications of Categories*, 14(17):399–423, 2005.

[20] D. Kruml and P. Resende. On quantales that classify C^*-algebras. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 45(4):287–296, 2004.

[21] H. Lai and L. Shen. Fixed points of adjoint functors enriched in a quantaloid. *Fuzzy Sets and Systems*, 321:1–28, 8 2017.

[22] F. W. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matématico e Fisico di Milano*, 43:135–166, 1973.

[23] F. W. Lawvere. Taking categories seriously. *Revista Colombiana de Matemáticas, XX*, pages 147–178, 1986.

[24] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 1998.

[25] B. Mesablishvili. Morita theory for quantales. *Journal of Algebra*, 689:764–784, 2026.

[26] H. Mori. Chu correspondences. *Hokkaido Mathematical Journal*, 37(1):147–214, 2008.

[27] C. J. Mulvey. &. *Supplemento ai Rendiconti del Circolo Matematico di Palermo Series II*, 12:99–104, 1986.

[28] A. L. T. Paterson. *Groupoids, Inverse Semigroups, and their Operator Algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser, Boston, 1999.

[29] J. W. Pelletier. Von Neumann algebras and Hilbert quantales. *Applied Categorical Structures*, 5(3):249–264, 1997.

[30] P. Resende. Quantales and observational semantics. In B. Coecke, D. Moore, and A. Wilce, editors, *Current Research in Operational Quantum Logic: Algebras, Categories, Languages*, pages 263–288. Springer, Dordrecht, 2000.

[31] P. Resende. Sup-lattice 2-forms and quantales. *Journal of Algebra*, 276(1):143–167, 2004.

[32] P. Resende. Étale groupoids and their quantales. *Advances in Mathematics*, 208:147–209, 1 2007.

[33] P. Resende. The many groupoids of a stably Gelfand quantale. *Journal of Algebra*, 498:197–210, 2018.

[34] P. Resende. Quantales and Fell bundles. *Advances in Mathematics*, 325:312–374, 2018.

[35] K. I. Rosenthal. *Quantales and their Applications*, volume 234 of *Pitman research notes in mathematics series*. Longman, Harlow, 1990.

[36] K. I. Rosenthal. *The Theory of Quantaloids*, volume 348 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996.

[37] L. Shen. *Adjunctions in Quantaloid-enriched Categories*. PhD thesis, Sichuan University, Chengdu, 2014.

[38] L. Shen, Y. Tao, and D. Zhang. Chu connections and back diagonals between $\mathcal{Q}$-distributors. *Journal of Pure and Applied Algebra*, 220(5):1858–1901, 2016.

[39] L. Shen and D. Zhang. Categories enriched over a quantaloid: Isbell adjunctions and Kan adjunctions. *Theory and Applications of Categories*, 28(20):577–615, 2013.

[40] I. Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory and Applications of Categories*, 14(1):1–45, 2005.

[41] S. Tull. Monoidal categories for formal concept analysis. *arXiv:2012.08268*, 2020.

[42] H. Wang. Ordinal sum of quantales. *Journal of Algebra*, 630:56–67, 2023.

[43] I. Weiss. Value semigroups, value quantales, and positivity domains. *Journal of Pure and Applied Algebra*, 223(2):844–866, 2019.